Characterizations of Dedekind domains

Atiyah-MacDonald

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2. If D has Krull dimension 1, then the $\mathfrak{p}_i$ are comaximal, so the Q_i are, so $Q_1 \cap \cdots \cap Q_r = Q_1 \cdots Q_r$.

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2. If D has Krull dimension 1, then the $\mathfrak{p}_i$ are comaximal, so the Q_i are, so $Q_1 \cap \cdots \cap Q_r = Q_1 \cdots Q_r$.
3. If D is also integrally closed, then primary = prime power, so $Q_i = \mathfrak{p}_i^{e_i}$, so $I = \mathfrak{p}_1^{e_i} \cdots \mathfrak{p}_r^{e_r}$.

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Preliminaries for the reverse direction

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DD implies prime factorization

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PID = UFD+DD

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