The ideal class group

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6. (Narkiewicz, 1964) If $\mathcal{O}_K$ has class number ≥ 3, then, for any positive k, almost every $\alpha \in \mathcal{O}_K$ has at least k different-length factorizations into irreducibles.

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This proves that if D is a Dedekind domain, then $[M] = [N]$ is equivalent to ${}_DM \cong {}_DN$.

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The structure of f.g. projective modules over a DD

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Proof.
[(1)$\Rightarrow$(2)]

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Proof.
[(1)$\Rightarrow$(2)] Integral ideals are f.g. and projective.

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Proof.
[(1)⇒(2)] Integral ideals are f.g. and projective. $\oplus$ preserves this.

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In this way, the ideal class group classifies all finitely generated projective D-modules.

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