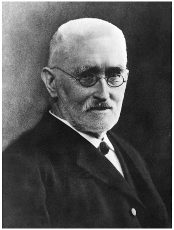
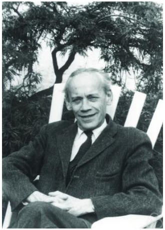
Chain conditions

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2. A nonempty poset P has the ACC+DCC iff every chain in P is finite.

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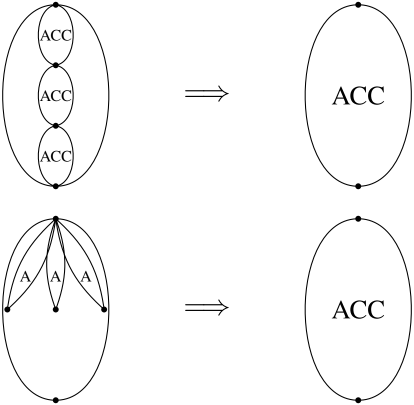

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All remain true if ‘Noetherian’ is replaced by ‘Artinian’.

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The ring $\mathbb{R}[x, y]$ may be viewed as the ring of polynomial functions from $\mathbb{R}^2 \to \mathbb{R}$.
Elements $g \in \mathbb{R}[x, y]$ can be restricted to Γ to obtain polynomial functions $g\colon \Gamma \to \mathbb{R}$. The ring of restricted functions is called the coordinate ring of Γ.
In this example, the coordinate ring $\mathbb{A}(\Gamma)$ is isomorphic to $\mathbb{R}[x, y]/(x^2 + y^2 - 1)$, since $g, h \in \mathbb{R}[x, y]$ have the same restriction to Γ iff $g - h$ restricts to the zero function on Γ iff $g - h$ is divisible by the irreducible polynomial $x^2 + y^2 - 1$.

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One last thing about Noetherian rings

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$A = \langle x_1, x_2, x_3, \ldots \mid x_1^2 = 0, x_2^2 = x_1, x_3^2 = x_2, \ldots \rangle$ has $\mathfrak{N} = (x_1, x_2, \ldots)$, and $\mathfrak{N}^2 = \mathfrak{N} \neq (0)$.

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3. Stage 3: Continue splitting off local factors. This must terminate in finitely many steps with A isomorphic to a product of Artinian local rings.

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Krull Height Theorem. If A is a commutative Noetherian ring and $I \lhd A$ can be generated by n elements, then any prime minimal over I has height at most n. (In particular, every prime has finite height.)

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Application. Let A be a commutative Noetherian ring. If $\mathfrak{p} < \mathfrak{q} < \mathfrak{r}$ are prime, then there exist infinitely may distinct primes $\mathfrak{q}_i$ such that $\mathfrak{p} < \mathfrak{q}_i < \mathfrak{r}$.

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