

COMMUTATIVE ALGEBRA

HOMEWORK ASSIGNMENT IV

PROBLEMS

All rings are commutative.

1. (Toby Aldape, Bob Kuo, Connor Meredith) Let $L, N \leq M$ be A -modules. Let U be the set of primes for which $L_{\mathfrak{p}} \subseteq N_{\mathfrak{p}}$ holds. Show that U is an intersection of open sets in $\text{Spec}(A)$. Show conversely that if V is any intersection of open sets in $\text{Spec}(A)$, then V is exactly the set of primes for which $L_{\mathfrak{p}} \subseteq N_{\mathfrak{p}}$ holds for some submodules L, N of some module M .

2. (Ezzedine El Sai, Michael Levet, Mateo Muro) Let M be an A -module, and let $m, n \in M$ and $r \in A$ be elements.

- (a) Show that the set of primes \mathfrak{p} where “ $m = 0$ ” is an open subset of $\text{Spec}(A)$, and that it is all of $\text{Spec}(A)$ iff $m = 0$. (Here $m \in N_{\mathfrak{p}}$ is shorthand for $\frac{m}{1} \in N_{\mathfrak{p}}$.)
- (b) Same type of problem for “ $m = n$ ”.
- (c) Same type of problem for “ r is nilpotent”.
- (d) Same type of problem for “ r is a unit”.

A ring is *subdirectly irreducible* (SI) if it has a least nonzero ideal. A module is SI if it has a least nonzero submodule.

3. (Howie Jordan, Chase Meadors, Adrian Neff) Prove in each of the following ways that an SI Noetherian ring must be Artinian:

- (a) Using primary decomposition: Show that A has a unique associated prime, which is a nilpotent maximal ideal. Then show that a Noetherian ring with a nilpotent maximal ideal is Artinian.
- (b) Using the Krull Intersection Theorem: again, first show that A has a nilpotent maximal ideal.

4. (Toby Aldape, Bob Kuo, Connor Meredith) Let M be a f.g. SI module over a Noetherian ring A .

- (a) Show that M is Artinian.

- (b) Show that M has a composition series, and that all composition factors are isomorphic.
5. (Ezzedine El Sai, Michael Levet, Mateo Muro)
- (a) Assume that A is Noetherian, that M is a finitely generated A -module and that $L, N \leq M$ are submodules. Show that $L \subseteq N$ iff $L_{\mathfrak{p}} \subseteq N_{\mathfrak{p}}$ holds for every $\mathfrak{p} \in \text{Ass}(M/N)$.
- (b) Show that any subset $U \subseteq \text{Spec}(A)$ can be $\text{Ass}(M)$ for some A -module M . Show that any finite subset $U_0 \subseteq \text{Spec}(A)$ can be the set of associated primes of some f.g. module.
6. (Howie Jordan, Chase Meadors, Adrian Neff)
- (a) Prove that if $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is exact, then $\text{Supp}(M) = \text{Supp}(L) \cup \text{Supp}(N)$.
- (b) Prove that $\text{Supp}(L \otimes_A N) = \text{Supp}(L) \cap \text{Supp}(N)$.
7. (Toby Aldape, Bob Kuo, Connor Meredith)
- (a) Let S be the set of elements of A that are not zero divisors. Show that S is the largest subset of A with the property that the canonical homomorphism $A \rightarrow S^{-1}A : a \mapsto a/1$ is an embedding. ($S^{-1}A$ is called the *total ring of fractions* of A .)
- (b) Show that if A is Noetherian, then the total ring of fractions of A has finitely many maximal ideals. (A ring with finitely many maximal ideals is called *semilocal*.) (Hint: Consider $\text{Ass}({}_A A)$.)
8. (Ezzedine El Sai, Michael Levet, Mateo Muro) Show that if A is an integrally closed domain and $f \in A[x]$ is a monic polynomial over A , then f irreducible over A iff F is irreducible over the field of fractions of A .
9. (Howie Jordan, Chase Meadors, Adrian Neff) Suppose that $A \leq B$ is an integral extension, and that B is finitely generated as an A -algebra. Show that for every prime $\mathfrak{p} \in \text{Spec}(A)$ there are only finitely many primes of B lying over \mathfrak{p} .