Algebraization

Example. (Euclidean space)

$$\begin{array}{ccccl} & \mathbb{E}^3 & \longrightarrow & \mathbb{R} & \longrightarrow \text{exact calculation} \\ \mathbb{E}^n = & \mathbb{R}^n & \longleftarrow & \mathbb{R} & \\ & \mathbb{F}^n & \longleftarrow & \mathbb{F} & \end{array}$$

Algebra

Definition. A 3-*sorted algebra* is a structure

$$\begin{aligned} & \langle A, B, C; x \oplus y, x \vee y, F_1(x), \ldots \rangle \\ = \quad & \langle \text{sets/universes/sorts; operations among sorts} \rangle \end{aligned}$$

Example. $\mathbb{Z} = \langle \text{integers}; \cdot, +, -, 0, 1 \rangle$.

Simplest examples of ‘algebraization’

Question. What are the laws of functional composition?
$(f \circ f) \circ (g \circ g) = (f \circ g) \circ (f \circ g)$?

Answer leads to

1. semigroups and monoids
2. groups
3. small categories
4. rings
5. k-algebras

1. What are we algebraizing?
2. How do we do it?
3. What did we learn?

Move to whiteboard!

Now we consider the composition of functions between different sets.

$\mathcal{C}$

$$
\begin{tikzcd}
A \arrow[drr, "g"] & & B \arrow[ll, "f"'] \arrow[d, "g \circ f"] \\
C \arrow[rr, "h"] & & D
\end{tikzcd}
$$

Definition. A *category* is a 2-sorted partial algebra $\mathcal{C} = \langle O, M; \circ, \mathrm{id}, \mathrm{dom}, \mathrm{cod} \rangle$ where

(1) $\mathrm{Ob}(\mathcal{C}) = O$ is a class whose members are called *objects*,
(2) $\mathrm{Mor}(\mathcal{C}) = M$ is a class whose members are called *morphisms*,
(3) $\circ : M \times M \to M$ is a binary partial operation called *composition*,
(4) $\mathrm{id} : O \to M$ is a unary function assigning to each object $A \in O$ a morphism id_A called the *identity* of A,
(5) $\mathrm{dom}, \mathrm{cod} : M \to O$ are unary functions assigning to each morphism f objects called the *domain* and *codomain* of f respectively.

The laws defining categories are:

(1) $f \circ g$ exists if and only if $\mathrm{dom}(f) = \mathrm{cod}(g)$.
(2) Composition is associative when it is defined.
(3) $\mathrm{dom}(f \circ g) = \mathrm{dom}(g)$, $\mathrm{cod}(f \circ g) = \mathrm{cod}(f)$.
(4) If $A = \mathrm{dom}(f)$ and $B = \mathrm{cod}(f)$, then $f \circ \mathrm{id}_A = f$ and $\mathrm{id}_B \circ f = f$.
(5) $\mathrm{dom}(\mathrm{id}_A) = \mathrm{cod}(\mathrm{id}_A) = A$.

Functors = homomorphisms between categories

Since categories are algebraic structures, we immediately know the meaning of *subcategory*, *quotient category*, etc., especially "homomorphism":

Definition. A *functor* $F : \mathcal{C} \to \mathcal{D}$ is a homomorphism from $\mathcal{C}$ to $\mathcal{D}$. In detail, F is a pair of mappings, both called F, between object classes and morphism classes, $F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ and $F\colon \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$, where

(1) $F(f \circ g) = F(f) \circ F(g)$,

(2) $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$,

(3) $F(\mathrm{dom}(f)) = \mathrm{dom}(F(f))$, and

(4) $F(\mathrm{cod}(f)) = \mathrm{cod}(F(f))$.

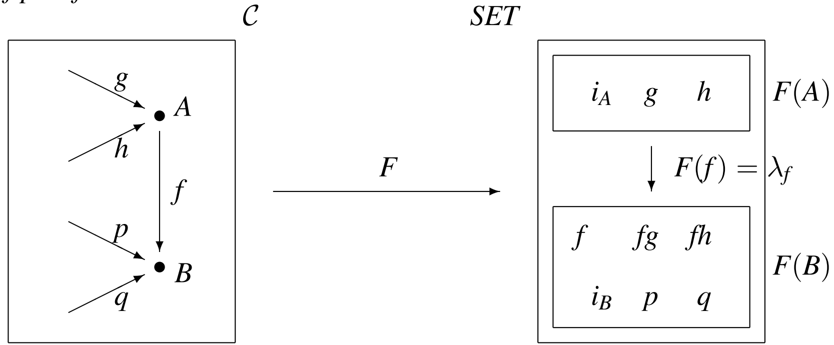
$\exists$ A Cayley Representation Theorem

A category is *small* if O and M are sets. (Enough to assume that M is a set.)

Theorem

Every small category is embeddable in the category of sets.

Idea of proof.



Examples of Categories

Examples.

(0) Empty category.

(1) (One object category) A category with one object is determined by its monoid of morphisms. If $\langle M; \circ, 1\rangle$ is a monoid, then

$$\mathcal{M} = \langle \{*\}, M; \circ, \mathrm{id}, \mathrm{dom}, \mathrm{cod}\rangle$$

is the associated category. Here $\mathrm{id}(*) = 1$ and $\mathrm{dom}, \mathrm{cod} : M \to \{*\}$ are constant. $M = \langle \text{ordinals}; +, 0\rangle$ yields a large 1-object category.

(2) If $\langle P; \leq\rangle$ is a partially ordered set, then the elements of P may be thought of as the objects of a category whose morphisms are the arrows $a \to b$ whenever $a \leq b$ in P. $P = \langle \text{ordinals}; \leq\rangle$ yields a large cat. of this type.

(3) Any class of algebras equipped with all algebra homomorphisms is a category. (E.g. **CRng**, R-**Mod**.)

(4) **Top** and **TopH** are categories.

Repeat

All of this can be repeated in the enriched category of abelian groups. This category is "enriched" in the sense that each hom-set $\mathrm{Hom}(A,B)$ has a structure beyond that of a set (i.e., it is an abelian group itself).

Reason:
If $\alpha, \beta \in \mathrm{Hom}(A,B)$, then the pointwise sum $(\alpha+\beta)(x) := \alpha(x) + \beta(x)$ is also in this set: compose

$$A \overset{(\alpha,\beta)}{\to} B \times B \overset{+}{\to} B$$

This is a hom, (say φ_+), since $+ : B \times B \to B$ is a hom. Check:

1. $\varphi_+((a,b)+(c,d)) = (a+c)+(b+d)$ and $\varphi_+((a,b)) + \varphi_+((c,d)) = (a+b)+(c+d)$.
2. $\varphi_+(-(a,b)) = (-a)+(-b)$ and $-\varphi_+((a,b)) = -(a+b)$.
3. $\varphi_+((0,0)) = 0+0$.
4. similarly $\varphi_-(x)$ and φ_0 are homs.

Repeat

If you repeat the algebraization arguments for $T(A)$ within the category of abelian groups, you will see that the correct algebraization of $\mathrm{End}(A)$ for A an abelian group is a structure

$$\langle R; \circ, +, -, 0, 1\rangle$$

such that

1. $\langle R; \circ, 1\rangle$ is a monoid
2. $\langle R; +, -, 0\rangle$ is an abelian group
3. distributive laws hold.

Conversely, the proof of the Cayley Repn Thm embeds any abstract structure satisfying these identities (i.e., any ring) into the concrete ring $\mathrm{End}(A)$.

If you replace the category of "abelian groups" with "vector spaces over field k" you obtain the definition of "k-algebra", which abstracts the properties of $\mathrm{End}_k(V)$, $M_n(k)$.

Why *commutative* rings?

1. (Number theory)
From the historical concept of "number":

$$\langle \mathbb{N}_+; S(x) \rangle \to \langle \mathbb{N}; \cdot, +, 0 \rangle \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{R} \to \mathbb{C}$$

2. (Functional analysis, algebraic geometry)
As auxiliary structures to study spaces.
E.g. for X a compact Hausdorff space X, the commutative ring $C(X)$ of continuous real-valued functions, plays a role similar to that played by the Galois group $\mathrm{Gal}(K/k)$, which is used to study the k-algebra structure of the extension field K.