Surprise Worksheet!

The goal of this worksheet is to prove the following theorem.

Theorem. Let P be an R-module. TFAE:

(1) P is projective.
(2) For any R-module M, if $\beta : M \to P$ is surjective, then β has a section.
(3) P is a direct summand of a free R-module.

Think about this, but there is no need to write down or turn in the solution.

Hints:

(For (1)⇒(2))
Apply the projective lifting property to

$$\begin{array}{ccccc} & & P & & \\ & & \big\downarrow{\scriptstyle id} & & \\ M & \xrightarrow{\beta} & P & \longrightarrow & 0 \end{array}$$

(For (2)⇒(3))
Explain why there is a free module and a surjective $\beta : F \to P$. Let δ be a section of β, and defined $\varepsilon = \delta \circ \beta$. Argue that ε is an idempotent endomorphism of F. This yields a kernel-image decomposition, $F \cong \ker(\varepsilon) \oplus \operatorname{im}(\varepsilon)$ with $\delta : P \to \operatorname{im}(\varepsilon)$ an isomorphism onto one of the summands.

(For (3)⇒(1))
Assume that F is free and that P is a direct summand of F. Then there exists Q and appropriate maps so that F is expressible as a biproduct
$(P \oplus Q, p_P, p_Q, \iota_P, \iota_Q)$.
Now suppose we must test the projectivity of P on $M \to N \to 0$ by lifting a map $\varphi : P \to N$ to a map $\varphi' : P \to M$ satisfying $\varphi = \beta \circ \varphi'$ in the diagram

$$\begin{array}{ccccc} & & P & & \\ & & \big\downarrow{\scriptstyle \varphi} & & \\ M & \xrightarrow{\beta} & N & \longrightarrow & 0 \end{array}$$

Approach this by considering the following diagram instead

$$\begin{array}{ccccc} & & F & & \\ & & \big\downarrow{\scriptstyle p_P} & & \\ & & P & & \\ & & \big\downarrow{\scriptstyle \varphi} & & \\ M & \xrightarrow{\beta} & N & \longrightarrow & 0 \end{array}$$

You should explain why there is a φ' such that $\varphi \circ p_P = \beta \circ \varphi'$. Right compose this equality of maps with ι_P to obtain $\varphi = \beta \circ (\varphi' \circ \iota_P)$.

Extension. Convince yourself of the truth of the following refinement:

A projective R-module P can be generated by $n + 1$ elements and not n elements if and only if it is a direct summand of an $(n + 1)$-generated free module but not a direct summand of an n-generated free module.