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Thm. If $\alpha : A \to B$ is a homomorphism and $\alpha(S) \subseteq B^\times$, then α has a unique extension to $\overline{\alpha} : S^{-1}A \to B$.

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If $S = \{f\}^* = \{1, f, f^2, \ldots\}$, then we often write A_f for $S^{-1}A$. These are viewed as rational functions of the form a/f^k defined on $\mathrm{Spec}(A_f) = \mathrm{Spec}(A) - V(f) = D(f) = \mathrm{supp}(f)$.

More generally, for any MC S, the ring $S^{-1}A$ is viewed as the ring of rational functions of the form a/s defined on on $\mathrm{Spec}(S^{-1}A) = \mathrm{Spec}(A) - \bigcup_{s \in S} V(s) = \bigcap_{s \in S} D(s) = \bigcap_{s \in S} \mathrm{supp}(s) = \mathrm{supp}(S)$.

An interesting special case is when $S = A - \mathfrak{p}$, where $\mathrm{Spec}(A_{\mathfrak{p}})$ equals the intersection of all distinguished open sets containing the point $\mathfrak{p}$.