COMMUTATIVE ALGEBRA
HOMEWORK ASSIGNMENT III

PROBLEMS

All rings are commutative.

1. (Howie Jordan, Bob Kuo, Mateo Muro)
Let M be a finitely generated R-module. Show that $R/I \otimes_R M = 0$ iff there exists $i \in I$ such that $(1+i) \in \mathbf{Ann}(M)$ in two different ways:

(1) using Nakayama's Lemma.
(2) avoiding Nakayama's Lemma.

2. (Ezzedine El Sai, Chase Meadors, Connor Meredith)
Let $\mathbb{F}$ be a field. Suppose that A and B are $\mathbb{F}$-algebras and that $B = \mathbb{F}[b]$ is generated as an $\mathbb{F}$-algebra by a single element $b \in B$.

(a) Show that $A \otimes_{\mathbb{F}} B \cong A[x]/\min_{b,\mathbb{F}}(x)$.
(b) Restrict now to the case where A and B are fields. Give an example where $A \otimes_{\mathbb{F}} B$ has nonzero nilpotent elements, and another example where $A \otimes_{\mathbb{F}} B$ $(\neq 0)$ has no nonzero nilpotent elements.

3. (Toby Aldape, Michael Levet, Adrian Neff)
Let m be an integer that is not a perfect square.

(a) Show that $\mathbb{Q}[\sqrt{m}] \otimes_{\mathbb{Q}} \mathbb{Q}[\sqrt{m}] \cong \mathbb{Q}[\sqrt{m}] \times \mathbb{Q}[\sqrt{m}]$ as $\mathbb{Q}$-algebras.
(b) Find the idempotents in $\mathbb{Q}[\sqrt{m}] \otimes_{\mathbb{Q}} \mathbb{Q}[\sqrt{m}]$ that induce the direct decomposition descibed in (a).
(c) Find an idempotent $e \neq 0, 1$ in $\mathbb{Q}[\sqrt[3]{2}] \otimes_{\mathbb{Q}} \mathbb{Q}[\sqrt[3]{2}]$.

4. (Howie Jordan, Bob Kuo, Mateo Muro)
(There is no contravariant analogue of the tensor product)

(a) Let k-Vec be the category of vector spaces over the field k. Show that the double dual functor $V \mapsto V^{**}$ is an additive covariant functor that is not representable.

(b) **A contravariant version of the tensor product, say $B \boxtimes_R C$, might be expected to satisfy the property that it represents the composite of the contravariant representable functors $\mathbf{Hom}_R(\_, B)$ and $\mathbf{Hom}_R(\_, C)$. Show that there is no such general construction for categories of modules.**

5. (Ezzedine El Sai, Chase Meadors, Connor Meredith)
Let $\varphi : R \to S$ be a ring homomorphism. Show that the functor of extension of scalars from R-Mod to S-Mod induces a commutative semiring homomorphism from $\langle R\text{-}\mathbf{Mod}; \otimes, \oplus, 0, R\rangle/\cong$ to $\langle S\text{-}\mathbf{Mod}; \otimes, \oplus, 0, S\rangle/\cong$.

6. (Toby Aldape, Michael Levet, Adrian Neff)
Suppose that

$$0 \longrightarrow \mathrm{Hom}_R(C, M) \xrightarrow{\circ\psi} \mathrm{Hom}_R(B, M) \xrightarrow{\circ\varphi} \mathrm{Hom}_R(A, M)$$

is exact for every R-module M. Show that

$$A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$$

is exact. (Hint: Consider what happens when $M = C/\mathrm{im}(\psi)$, $M = C$ and $M = B/\mathrm{im}(\varphi)$.)

7. (Howie Jordan, Bob Kuo, Mateo Muro)
An R-module M is finitely presentable iff there is an exact sequence

$$\oplus^n R \xrightarrow{\alpha} \oplus^m R \xrightarrow{\beta} M \longrightarrow 0.$$

Show that if $0 \to A \to B \to C \to 0$ is exact, and both A and C are finitely presentable, then B is finitely presentable.

8. (Ezzedine El Sai, Chase Meadors, Connor Meredith)
Show that if A is a flat R-module, then its character module $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ is an injective R-module. (The converse is also true, but you don't have to prove it.)

9. (Toby Aldape, Michael Levet, Adrian Neff)
Show that the the class $\mathcal{P}$ of projective R-modules is closed under $\oplus$ and $\otimes$ and contains 0 and R. Show that the same is true if we replace $\mathcal{P}$ with the subclass $\mathcal{P}_{\mathbf{f.g.}}$ of finitely generated projective R-modules.