COMMUTATIVE ALGEBRA
HOMEWORK ASSIGNMENT II

Read Chapter 2

PROBLEMS

All rings are commutative.

1. (Bob Kuo, Ezzeddine El Sai, Adrian Neff) Classify all the maximal subrings of $\mathbb{Q}$, and show that any two of them have homeomorphic spectra.

2. (Toby Aldape, Mateo Muro, Chase Meadors)
(Nilradical versus Jacobson radical)

(a) **Show that $\mathfrak{N}(R \times S) = \mathfrak{N}(R) \times \mathfrak{N}(S)$ and $J(R \times S) = J(R) \times J(S)$. Hence, if the nilradical and the Jacobson radical are equal in each coordinate of a product, then they are equal in the product.**

(b) **Show the result of part (a) does not hold for infinite products by showing that the nilradical and Jacobson radical are equal in all coordinates of the product $T = \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8 \times \cdots$, but $\mathfrak{N}(T) \neq J(T)$.**

3. (Connor Meredith, Michael Levet, Howie Jordan) Prove that the Jacobson radical contains no nonzero idempotents in each of the following ways:

(a) **using the characterization of $J(R)$ as the intersection of maximal ideals.**

(b) **using the characterization of $J(R)$ as the largest ideal J such that $1 + J$ consists of units.**

(c) **using the characterization of $J(R)$ as the intersection of annihilators of all simple modules.**

4. (Bob Kuo, Ezzeddine El Sai, Adrian Neff) Show that $J(R)$ and $\mathfrak{N}(R)$ can be characterized in the following ways.

(a) **$J(R)$ is the largest ideal $J \lhd R$ such that all covers below J in Ideal(R) are of abelian type. (That is, $I \prec K \leq J$ implies $K^2 \subseteq I$.)**

(b) $\mathfrak{N}(R)$ is the largest ideal $I \lhd R$ such that there is a well-ordered chain of ideals

$$0 = I_0 \leq I_1 \leq I_2 \leq \cdots \leq I_\mu = I$$

such that

(i) $I_{\alpha+1}$ is abelian over I_α for all α, and

(ii) $I_\lambda = \bigcup_{\kappa<\lambda} I_\kappa$ whenever λ is a limit ordinal.

5. (Toby Aldape, Mateo Muro, Chase Meadors) Suppose that $I \lhd R$ has infinitely many primes that are minimal above it.

(a) Show that I is not prime.

(b) Use (a) to show that there is an ideal properly containing I that also has infinitely many minimal primes above it.

(c) Conclude that R is not Noetherian. (Expressed more positively, any Noetherian ring has the property that every ideal I has only finitely many minimal primes containing it, hence $\sqrt{I}$ is an intersection of finitely many primes.)

6. (Connor Meredith, Michael Levet, Howie Jordan) Here we consider Spec(R) as a topological space (the primes equipped with the Zariski topology), and as an ordered set (the primes equipped with the inclusion order).

(a) Show that the inclusion order on the prime ideals can be recovered from the topology of Spec(R).

(b) Show that conversely, if R is a Noetherian ring, then the topology of Spec(R) can be determined from the inclusion order on the prime ideals.

(c) Show that if R is not Noetherian, then the topology of Spec(R) may not be recoverable from the inclusion order on the primes.

7. (Bob Kuo, Ezzeddine El Sai, Adrian Neff)

(a) Suppose that R is a UFD. Show that a prime ideal in R is generated as an ideal by the irreducible elements it contains.

(b) Now suppose that $R = S[x]$ where S is a PID. Show that any prime ideal of R is generated by at most 2 irreducible elements. Show that if a prime ideal requires two irreducible generators, then it has the form $I = (p, f(x))$ where p is prime in S and $f(x)$ is a monic polynomial in $S[x]$ that is irreducible mod p.

(c) (continued from (b)) Sketch the ordered set of primes of $S[x]$ under inclusion to the best of your ability. How long can a chain be?

8. (Toby Aldape, Mateo Muro, Chase Meadors)

(a) **Show that $J(M)$ consists of the *nongenerators* of M: i.e., $m \in J(M)$ iff $M = \langle S \cup \{m\} \rangle$ implies $M = \langle S \rangle$. (This means single elements of $J(M)$ may be cancelled from any generating set.)**

(b) **Exhibit an example to show that it infinitely many elements from $J(M)$ might not be cancellable from a generating set.**

(c) **Show that if M is finitely generated and $P \subseteq J(M)$, then $M = N + P$ implies $M = N$. (This means any set of elements of $J(M)$ may be cancelled from a generating set of a finitely generated module.) In particular, show that if $I \subseteq J(R)$, M is finitely generated, and $M = N + IM$, then $M = N$.**

9. (Connor Meredith, Michael Levet, Howie Jordan) This problem involves Nakayama's Lemma.

Suppose that $(R, \mathfrak{m})$ is a local ring with maximal ideal $\mathfrak{m}$, and that M is a finitely generated R-module.

(a) **Show that a subset $F \subseteq M$ is a generating set iff $F/\mathfrak{m}$ is a generating set for the $R/\mathfrak{m}$-vector space $M/\mathfrak{m}M$. Conclude that all minimal generating sets for M have the same size.**

(b) **Show that a homomorphism of $\varphi\colon M \to N$ between finitely generated R-modules is surjective iff the induced map $\varphi_{\mathfrak{m}}\colon M/\mathfrak{m}M \to N/\mathfrak{m}N$ is surjective. (Your solution should say why there is an "induced map".)**