COMMUTATIVE ALGEBRA
HOMEWORK ASSIGNMENT I

Read Chapter 1

PROBLEMS

All rings are commutative.

1. (Bob Kuo, Michael Levet, Chase Meadors) Let k be a field. Describe the ideal lattices of

(a) $k[x]$.
(b) $k[[x]]$ **(= the ring of formal power series over k, $f = \sum_{n=0}^{\infty} a_n x^n$).**
(c) $k((x))$ **(= the ring of Laurent series over k, $f = \sum_{n=r}^{\infty} a_n x^n$, $r \in \mathbb{Z}$).**

In each case, specify which ideals are maximal or prime.

2. (Toby Aldape, Ezzeddine El Sai, Howie Jordan) Let R be an integral domain and let K be its field of fractions. Show that the following are equivalent.

(a) For every $x \in K$, either $x \in R$ or $x^{-1} \in R$.
(b) The ideal lattice of R is a chain.

3. (Connor Meredith, Mateo Muro, Adrian Neff)

(a) Show that a ring R is directly decomposable as a ring iff it is directly decomposable when considered as an R-module.
(b) Show that an R-module M is directly decomposable iff it has an idempotent endomorphism $\varepsilon\colon M \to M$ such that $\ker(\varepsilon) \neq 0 \neq \mathrm{im}(\varepsilon)$.
(c) Show that the R-module endomorphisms of ${}_RR$ all have the form $\varepsilon(x) = rx$ for some $r \in R$.
(d) Show that any direct decomposition of R has the form $R \cong R/(e) \times R/(1-e)$ for some idempotent $e \in R$.

4. (Bob Kuo, Michael Levet, Chase Meadors) Show that the ideals of $R \times S$ are of the form $I \times J$ where $I \lhd R$ and $J \lhd S$. Show that the prime (maximal) ideals have the form $P \times S$ and $R \times Q$ for prime (maximal) ideals $P \lhd R$ and $Q \lhd S$.

5. (Toby Aldape, Ezzeddine El Sai, Howie Jordan) Suppose that $I \lhd R$ is a nil ideal (meaning: every element of I is nilpotent).

(a) Show that $a + I$ is a unit in R/I iff a is a unit in R.

(b) Show that $a + I$ is idempotent in R/I iff there exists an idempotent $e \in R$ such that $e + I = a + I$. (Idempotents can be lifted modulo a nil ideal.) (Hint for "only if": use the fact that $[a(1-a)]^n = 0$ for some n, then expand $(a + (1-a))^{2n}$.)

6. (Connor Meredith, Mateo Muro, Adrian Neff) Let I be a minimal nonzero ideal of the commutative ring R.

(a) Show that $(0 : I)$ is a maximal ideal.

(b) Show that if $I^2 = I$, then $(0 : I)$ is a complement to I and $R \cong R/I \times R/(0 : I)$.

7. (Bob Kuo, Michael Levet, Chase Meadors) A *chain* of ideals is a set of ideals linearly ordered by $\subseteq$.

(a) Show that if $(P_i)_{i \in I}$ is a chain of primes, then $\bigcup P_i$ and $\bigcap P_i$ are primes.

(b) Show that if I is an ideal contained in a prime ideal P, then there is a prime ideal P' such that $I \subseteq P' \subseteq P$ and P' is "minimal prime over I" (meaning that there is no prime P'' satisfying $I \subseteq P'' \subsetneq P'$).

(c) Show that if I is an ideal containing a prime ideal P, then there is a prime ideal P' such that $I \supseteq P' \supseteq P$ and P' is "maximal prime under I".

8. (Toby Aldape, Ezzeddine El Sai, Howie Jordan) Explain why any commutative ring is a homomorphic image of a subring of a field. Conclude that commutative rings satisfy all positive universal[1] sentences true in all fields. Explain how this shows that (for example) the truth of the Cayley-Hamilton Theorem for fields implies the truth of this theorem for any commutative ring.

9. (Connor Meredith, Mateo Muro, Adrian Neff) Show that the map $\mathbf{Idl}(R) \to L \mapsto I \mapsto \mathrm{nil}(I)$ from the lattice of ideals of R to the lattice of semiprime ideals is a homomorphism with respect to binary $\wedge$ and infinitary $\bigvee$.

[1]A sentence, "$Q_1 x_1 \cdots Q_n x_n$(quantifier-free part)", in an algebraic language, where the Q's are quantifiers, is *positive* if the quantifier-free part is built up from equations using only "and" and "or" and is *universal* if the quantifiers are all universal quantifiers ($\forall$).