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9. Suppose that $A \leq B$ is an integral extension, and that B is finitely generated as an A-algebra. Show that for every prime $\mathfrak{p} \in \operatorname{Spec} A$ there are only finitely many primes of B lying over $\mathfrak{p}$.

Lemma: If R is a finite algebra (finitely generated as a module) over a field k, then R is Artinian.

Proof of Lemma: Let $I_1 \supseteq I_2 \supseteq \ldots$ be a descending chain of ideals in R. R is a vector space over k, so each I_j is actually a k-subspace of R. Thus this is actually a descending chain of subspaces in a finite dimensional vector space, and therefore must stabilize as the dimension must stop decreasing. Hence R is an Artin ring. $\square$

Proof of Problem 9: Let $\mathfrak{p} \in \operatorname{Spec} A$ and let $\mathfrak{q} \in \operatorname{Spec} B$ be any prime lying over $\mathfrak{p}$. We have the following commutative diagram, with int indicating an integral extension:

$$\begin{array}{ccc} A & \xrightarrow{\text{int}} & B \\ \downarrow & & \downarrow \\ A_{\mathfrak{p}} & \xrightarrow{\text{int}} & B_{\mathfrak{p}} \end{array}$$

First, since B is a finitely generated A-algebra, $B_{\mathfrak{p}}$ will be a finitely generated $A_{\mathfrak{p}}$-algebra. The bottom map that expresses $B_{\mathfrak{p}}$ as an $A_{\mathfrak{p}}$-algebra is an integral extension, so $B_{\mathfrak{p}}$ must actually be a finitely generated $A_{\mathfrak{p}}$-module (proof in Atiyah-MacDonald that "finite-type + integral = finite", page 60).

Next, along the left arrow, $\mathfrak{p}$ maps to $\mathfrak{p}A_{\mathfrak{p}}$, which is the unique maximal ideal in $A_{\mathfrak{p}}$. We will denote this by $\mathfrak{m}_{\mathfrak{p}}$. Let $\mathfrak{q}'$ be the image of $\mathfrak{q}$ in $B_{\mathfrak{p}}$. Now under the bottom arrow, we must have $(\mathfrak{q}')^c = \mathfrak{m}_{\mathfrak{p}}$, which implies that $\mathfrak{q}'$ is maximal (it contracts to a maximal ideal by an integral extension). Since $(\mathfrak{q}')^c = \mathfrak{m}_{\mathfrak{p}}$, we must have $\mathfrak{q}' \supseteq \mathfrak{m}_{\mathfrak{p}}B_{\mathfrak{p}}$. Thus $\mathfrak{q}'$ will correspond to a maximal ideal in $B_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}B_{\mathfrak{p}}$. $\mathfrak{m}_{\mathfrak{p}}$ annihilates $B_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}B_{\mathfrak{p}}$, so $B_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}B_{\mathfrak{p}}$ is actually a finitely generated module over $A_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}$, i.e., it is a finite dimensional vector space over this field. Therefore, by the lemma, $B_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}B_{\mathfrak{p}}$ is Artinian, and thus has only finitely many maximal ideals. Hence, any $\mathfrak{q} \in \operatorname{Spec} B$ lying over $\mathfrak{p}$ corresponds to a maximal ideal in $B_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}B_{\mathfrak{p}}$, and no two such prime ideals will correspond to the same maximal ideal, so there must be only finitely many primes of B lying over $\mathfrak{p}$. $\square$