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Commutative Algebra
Assignment 4

1. Show that if A is an integrally closed domain and $f \in A[x]$ is a monic polynomial over A, then f irreducible over A iff f is irreducible over the field of fractions of A.

Proof. Let k be the field of fractions of A. We prove the equivalent statement that says f is reducible in A iff f is reducible over the field of fractions of A.

Suppose f is monic and reducible in $A[x]$. There exist nontrivial factorization $f = gh$ for $g, h \in A[x]$. g and h can be taken to be monic. In particular, since the factorization is nontrivial, they must both be of nonzero degree. Then both must be of degree strictly less than f. Since $A[x]$ is a subring of $k[x]$, then $g, h \in k[x]$. Hence gh is a nontrivial factorization of f in $k[x]$.

Suppose that $f = gh$ for $g, h \in k[x]$. We may assume g and h are monic. There exists an algebraically closed field $\overline{k}$ containing k in which the two polynomials can be factored as

$$g = \prod^{k}(x - \gamma_i) \quad h = \prod^{l}(x - \eta_i).$$

The roots of g and h in $\overline{k}$ are roots of f, hence they are integral over A. Each coefficient of g or h belongs to the ring generated by these roots, hence the coefficients of g and h are integral over A. Since the coefficients of g and h belong to k and they are integral over A, they belong to A. This shows that $f = gh$ is a factorization over A.

$\square$