Toby Aldape
Bob Kuo
Connor Meredith

7. Let A be a commutative ring with 1.

(a) Let S be the set of elements of A that are not zero divisors. Show that S is the largest subset of A with the property that the canonical homomorphism $A \to S^{-1}A : a \mapsto a/1$ is an embedding. ($S^{-1}A$ is called the *total ring of fractions* of A.

(b) Show that if A is Noetherian, then the total ring of fractions of A has finitely many maximal ideals. (A ring with finitely many maximal ideals is called *semilocal*.)

(a) We first show that S satisfies the desired property. For this purpose, let $\varphi : A \to S^{-1}A : a \mapsto a/1$ be the canonical homomorphism and suppose $a, b \in A$ with $\varphi(a) = \varphi(b)$. Then by definition of $S^{-1}A$, there is some $u \in S$ such that $u(a \cdot 1 - b \cdot 1) = 0$. Since u is not a zero divisor, we must have $a \cdot 1 - b \cdot 1 = 0$, or simply $a = b$, so φ is injective.

Now suppose T is a multiplicatively closed subset of A that contains a zero divisor, say z, and let $\phi : A \to T^{-1}A, a \mapsto a/1$ be the canonical homomorphism. Since z is a zero divisor, there is some nonzero $w \in A$ such that $zw = 0$. Rewriting this, we have $z(1 \cdot w - 0 \cdot 1) = 0$, so $\phi(w) = \phi(0)$. However, w was nonzero, so this shows that ϕ is not injective. Overall, we see that if T is any multiplicatively closed subset of A containing a zero divisor, then $A \to T^{-1}A : a \mapsto a/1$ is not an embedding.

By the previous paragraph, if T is a multiplicatively closed subset of A such that $A \to T^{-1}A : a \mapsto a/1$ *is* an embedding, then T must not contain any zero divisors, and therefore must be contained in S.

(b) We first prove two lemmas. The second lemma will show that $\mathrm{Ass}({}_AA)$ is finite, which we will use to show that $S^{-1}A$ has only finitely many maximal ideals. The first lemma will be used several times in proving the second.

Lemma 1: If $p \in \mathrm{Ass}({}_AA)$, then $\mathrm{Ass}({}_AA/p) = \{p\}$.

Proof: Suppose $p \in \mathrm{Ass}({}_AA)$ and recall that $p = (0 : m)$ for some $m \in A$ and $A/p \cong \langle m \rangle$. This immediately yields $p \in \mathrm{Ass}({}_AA/p)$. Now suppose $q \in \mathrm{Ass}({}_AA/p)$. We will show that $q = p$.

- Since $q \in \mathrm{Ass}({}_AA/p)$, there is some $n \in A - p$ such that $q = (0 : n/p)$. That is, $q = \{x \in A \mid xn \in p\}$. This yields $q \subseteq \{x \in A \mid x \in p\} = p$ once we recognize that p is prime and $n \notin p$.

- Choose $m \in A$ so that $A/p \cong \langle m \rangle = \{rm \mid r \in A\}$ and let $n \in \langle m \rangle$ so that $q = (0 : n)$. Since $n \in \langle m \rangle$, there is some $r \in A$ such that $rm = n$. Now let $x \in p$. Then

$$\begin{aligned} x \cdot n &= x \cdot (rm) \\ &= x \cdot (r \cdot m) \\ &= (xr) \cdot m \\ &= (rx) \cdot m \\ &= r \cdot (x \cdot m) \\ &= r \cdot 0 \\ &= 0 \end{aligned}$$

  so $x \in q$. Ultimately, $p \subseteq q$.

These two points together show $q = p$, so in fact $\mathrm{Ass}({}_A A/p) = \{p\}$. $\square$

Lemma 2: If M is a finitely generated module over a Noetherian ring A, then $\mathrm{Ass}(M)$ is finite.

Proof: If $M = 0$ we are done, so suppose $M \neq 0$. Suppose for sake of contradiction that $\mathrm{Ass}(M)$ is infinite and let $p_1, p_2, ...$ be infinitely many distinct elements of $\mathrm{Ass}(M)$. We will inductively build an infinitely ascending chain of submodules $M_0 < M_1 < M_2 < \cdots$ of M such that for all $i \in \mathbb{N}$, $M_{i+1}/M_i \cong A/p_{i+1}$ and $\mathrm{Ass}(M_i) \subseteq \{p_1, ..., p_i\}$. Since M is finitely generated and A is Noetherian, M is itself Noetherian, so this will be a contradiction. To begin, set $M_0 = 0$ and recall that associated primes are prime annihilators. In particular, there is some $m_1 \in M$ such that $A/p_1 \cong \langle m_1 \rangle \leq M$. Let $M_1 = \langle m_1 \rangle$. We see directly that $M_1/M_0 = M_1 \cong A/p_1$ and $\mathrm{Ass}(M_1) = \mathrm{Ass}(A/p_1) = \{p_1\}$.

Let $i > 1$ and suppose the chain $M_0 < M_1 < \cdots < M_i$ has already been constructed. There is an exact sequence of modules $0 \to M_i \to M \to M/M_i \to 0$, which yields the containments

$$\mathrm{Ass}(M_i) \subseteq \mathrm{Ass}(M) \subseteq \mathrm{Ass}(M_i) \cup \mathrm{Ass}(M/M_i)$$

By the inductive hypothesis, $\mathrm{Ass}(M_i) \subseteq \{p_1, ..., p_i\}$, which does not contain p_{i+1}. Since $p_{i+1} \in \mathrm{Ass}(M) \subseteq \{p_1, ..., p_i\} \cup \mathrm{Ass}(M/M_i)$, we therefore have $p_{i+1} \in \mathrm{Ass}(M/M_i)$. Hence, there is some $m_{i+1} \in M/M_i$ with $A/p_{i+1} \cong \langle m_{i+1} \rangle \leq M/M_i$. Let M_{i+1} be the unique submodule of M properly containing M_i such that $M_{i+1}/M_i = \langle m_{i+1} \rangle$.

It remains to check that $M_{i+1}/M_i \cong A/p_{i+1}$ and $\mathrm{Ass}(M_i) \subseteq \{p_1, ..., p_{i+1}\}$. For the former condition, note that by construction, $M_{i+1}/M_i = \langle m_{i+1} \rangle \cong A/p_{i+1}$. For the latter condition, note that there is a short exact sequence $0 \to M_i \to M_{i+1} \to M_{i+1}/M_i \to 0$. This sequence yields the containments $\mathrm{Ass}(M_i) \subseteq \mathrm{Ass}(M_{i+1}) \subseteq \mathrm{Ass}(M_i) \cup \mathrm{Ass}(M_{i+1}/M_i)$. By inductive hypothesis, $\mathrm{Ass}(M_i) \subseteq \{p_1, ..., p_i\}$, so we have

$$\mathrm{Ass}(M_{i+1}) \subseteq \{p_1, ..., p_i\} \cup \mathrm{Ass}(M_{i+1}/M_i) = \{p_1, ..., p_i\} \cup \mathrm{Ass}(A/p_{i+1}) = \{p_1, ..., p_{i+1}\}$$

$\square$

An immediate corollary of the lemma is that $\mathrm{Ass}({}_A A)$ is finite: A is a Noetherian ring and is therefore finitely generated as a left A-module under the usual action. We intend to use this to show that $S^{-1}A$ only has finitely many maximal ideals. We will show that the maximal primes of $S^{-1}A$ are in bijective correspondence with the primes of A maximal for the property of being contained in a (finite) union of associated primes. Then by appealing to prime avoidance, we conclude that such primes are simply the primes appearing in the union, of which there are only finitely many, completing the proof.

Recall that the primes of $S^{-1}A$ are in bijective, order-preserving, correspondence with the primes of A disjoint from S. Since S is the set of all non-zero-divisors of A, the primes of $S^{-1}A$ are in bijective, order preserving, correspondence with the primes of A consisting entirely of zero divisors.

Let m be a nonzero zero divisor of A and suppose $x \in (0 : m)$. If $x \neq 0$, then x is itself a zero divisor since $xm = 0$. Hence, we can write the set of zero divisors of A as the union of all annihilators of nonzero zero divisors. Or, even better, we can write the set of zero divisors of A as the union of all maximal elements of $\mathcal{S} := \{(0 : m) \mid m \in A - \{0\}\}$. We now show that every maximal element of $\mathcal{S}$ is an associated prime of ${}_A A$. Since associated primes are prime annihilators, it suffices to show maximal elements of $\mathcal{S}$ are prime. Let $(0 : m)$ be maximal in S and suppose $rs \in (0 : m)$ with $s \notin (0 : m)$. Since A is commutative, any annihilator of m annihilates sm. Therefore, $(0 : m) \subseteq (0 : sm)$. Since $s \notin (0 : m)$, $sm \neq 0$ and so $(0 : sm) \in \mathcal{S}$. But $(0 : m)$ is maximal and below $(0 : sm)$, so $(0 : sm) = (0 : m)$. Moreover, since $rs \in (0 : m)$, we have $rsm = 0$, so $r \in (0 : sm)$. Hence, $r \in (0 : m)$, so $(0 : m)$ is prime. Since $\mathrm{Ass}({}_A A)$ is finite, we can now list the maximal elements of $\mathcal{S}$ as p_1, ..., p_k.

The previous two paragraphs together imply that the primes of $S^{-1}A$ are in bijective, order-preserving, correspondence with the primes of A contained in $p_1 \cup \cdots \cup p_k$. In particular, the maximal ideals of $S^{-1}A$ (which are prime) are in bijective correspondence with the primes maximal for the property of being contained in $p_1 \cup \cdots \cup p_k$. By the prime avoidance lemma, every ideal contained in $p_1 \cup \cdots \cup p_k$ is contained in p_i for some $i \in \{1, ..., k\}$. Hence, a prime maximal for the property of being contained in $p_1 \cup \cdots \cup p_k$ is a prime maximal for the property of being contained in some p_i, i.e., is equal to some p_i. Overall, we see that $S^{-1}A$ has k maximal ideals, so we are done.