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6.

(a) Prove that if $0 \to L \to M \to N \to 0$ is exact, then $\text{Supp}(M) = \text{Supp}(L) \cup \text{Supp}(N)$.

(b) Prove that $\text{Supp}(L \otimes_A N) = \text{Supp}(L) \cap \text{Supp}(N)$.

Proof.

(a) Let L, M, N be A modules for some commutative ring A, and let

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

be an exact sequence. Recall that the support of an A module X is the set of all prime ideals $\mathbf{p}$ of A for which the localization of X at $\mathbf{p}$ is nonzero. We will show the equivalent statement that $\text{Supp}(M)^c = \text{Supp}(L)^c \cap \text{Supp}(N)^c$, where $\text{Supp}(X)^c$ is the complement of $\text{Supp}(X)$ in $\text{Spec}(A)$, that is, the set of all primes $\mathbf{p}$ for which $X_{\mathbf{p}} = 0$.

We start by showing $\text{Supp}(M)^c \subseteq \text{Supp}(L)^c \cap \text{Supp}(N)^c$. Suppose that $\mathbf{p} \in \text{Supp}(M)^c$. Since localization is an exact functor[1] we have that

$$0 \longrightarrow L_{\mathbf{p}} \xrightarrow{f_{\mathbf{p}}} M_{\mathbf{p}} \xrightarrow{g_{\mathbf{p}}} N_{\mathbf{p}} \longrightarrow 0$$

is also exact. Since we chose $\mathbf{p}$ from the complement of the support of M, $M_{\mathbf{p}} = 0$ so the sequence reduces to

$$0 \longrightarrow L_{\mathbf{p}} \xrightarrow{f_{\mathbf{p}}} 0 \xrightarrow{g_{\mathbf{p}}} N_{\mathbf{p}} \longrightarrow 0.$$

Exactness at L_p means that $f_{\mathbf{p}}$ is injective and hence $\ker(f_{\mathbf{p}}) = 0$. However, $M_{\mathbf{p}} = 0$ implies that $f_{\mathbf{p}}$ is the 0 map, so that $\ker(f_{\mathbf{p}}) = L_p$. Hence, $L_{\mathbf{p}} = 0$ so that $\mathbf{p} \in \text{Supp}(L)^c$. Similarly, exactness at $N_{\mathbf{p}}$ and $M_{\mathbf{p}} = 0$ implies that $N_{\mathbf{p}} = 0$ so that $\mathbf{p} \in \text{Supp}(N)^c$. Hence $\mathbf{p} \in \text{Supp}(L)^c \cap \text{Supp}(N)^c$.

Now we show $\text{Supp}(L)^c \cap \text{Supp}(N)^c \subseteq \text{Supp}(M)^c$. Let $\mathbf{p} \in \text{Supp}(L)^c \cap \text{Supp}(N)^c$. Then $L_{\mathbf{p}} = 0$ and $N_{\mathbf{p}} = 0$. Exactness of localization again implies that the above sequence of localizations is exact, however now we have

$$0 \longrightarrow 0 \xrightarrow{f_{\mathbf{p}}} M_{\mathbf{p}} \xrightarrow{g_{\mathbf{p}}} 0 \longrightarrow 0$$

and so as above, exactness at $M_{\mathbf{p}}$ implies that $M_{\mathbf{p}} = 0$. Hence $\mathbf{p} \in \text{Supp}(M)^c$.

We have shown bicontainment, so we have that $\text{Supp}(M)^c = \text{Supp}(L)^c \cap \text{Supp}(N)^c$, from which the claim follows.

[1] Stacks Project, Prop. 10.9.12, https://stacks.math.columbia.edu/tag/00CS.

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(b) Let L and N be A modules for a commutative ring A. First we will show that the containment $\mathrm{Supp}(L \otimes_A N) \subseteq \mathrm{Supp}(L) \cap \mathrm{Supp}(N)$ holds. However, the containment $\mathrm{Supp}(L) \cap \mathrm{Supp}(N) \subseteq \mathrm{Supp}(L \otimes_A N)$ is not true in general and we provide a counterexample. Taking L and N to be finitely generated, we are able to establish the second containment.

Suppose that $\mathbf{p} \in \mathrm{Supp}(L \otimes_A N)$. We will use the fact that a prime $\mathbf{p}$ is in the support of an A module X if and only if $\mathbf{p}$ contains the annihilator of X. This follows from Stacks Project Lemma 10.39.5[2] which shows that $V(\mathrm{Ann}(X)) = \mathrm{Supp}(X)$. Hence, we have that $\mathrm{Ann}(L \otimes_A N) \subseteq \mathbf{p}$. We wish to show $\mathbf{p} \in \mathrm{Supp}(L) \cap \mathrm{Supp}(N)$, and it suffices to show that $\mathrm{Ann}(L) \cup \mathrm{Ann}(N) \subseteq \mathbf{p}$, i.e. that $\mathbf{p}$ contains the annihilators of both L and N. If $r \in \mathrm{Ann}(L)$, then $r \cdot (l \otimes n) = r \cdot l \otimes n = 0 \otimes n = 0$ for all simple tensors $l \otimes n \in L \otimes_A N$, hence $r \in \mathrm{Ann}(L \otimes_A N)$. Similarly, if $r \in \mathrm{Ann}(N)$ we have that r annihilates all simple tensors so that $r \in \mathrm{Ann}(L \otimes_A N)$. So we have that $\mathrm{Ann}(L) \cup \mathrm{Ann}(N) \subseteq \mathrm{Ann}(L \otimes_A N) \subseteq \mathbf{p}$. Since $\mathbf{p}$ contains the annihilators of both L and N, it is in the support of both, so that $\mathbf{p} \in \mathrm{Supp}(L) \cap \mathrm{Supp}(N)$.

To see that the reverse containment does not hold in general, note the following case. Take the ring $A = \mathbb{Z}$ and the modules $L = \mathbb{Q}$ and $N = \mathbb{Z}/2\mathbb{Z}$. Then $L \otimes_A N = 0$, as for any simple tensor $\frac{a}{b} \otimes z \in L \otimes_A N$ we have

$$\frac{a}{b} \otimes z = 2(\frac{a}{2b} \otimes z) = \frac{a}{2b} \otimes 2z = \frac{a}{2b} \otimes 0 = 0.$$

So, it must be that $\mathrm{Supp}(L \otimes_A N) = \varnothing$. However, we will show that $\mathrm{Supp}(L) \cap \mathrm{Supp}(N) \neq \varnothing$ so that we do not have $\mathrm{Supp}(L) \cap \mathrm{Supp}(N) \subseteq \mathrm{Supp}(L \otimes_A N)$.

The annihilator of L as a $\mathbb{Z}$ module is the 0 ideal, hence all prime ideals of $\mathbb{Z}$ contain $\mathrm{Ann}(L)$ and $\mathrm{Supp}(L) = \mathrm{Spec}(\mathbb{Z})$. For N, recall from above that $\mathbf{p} \in \mathrm{Supp}(N)$ if and only if $\mathrm{Ann}(N) \subseteq \mathbf{p}$. The $\mathrm{Ann}(\mathbb{Z}/2\mathbb{Z}) = 2\mathbb{Z}$, hence the maximal ideal $2\mathbb{Z}$ is the only prime in $\mathrm{Supp}(N)$. But then, $\mathrm{Supp}(L) \cap \mathrm{Supp}(N) = \{2\mathbb{Z}\}$.

We now take L and N to be finitely generated, and demonstrate in this case that $\mathrm{Supp}(L) \cap \mathrm{Supp}(N) \subseteq \mathrm{Supp}(L \otimes_A N)$. Take some $\mathbf{p} \in \mathrm{Supp}(L) \cap \mathrm{Supp}(N)$. We will construct a surjective map from $(L \otimes_A N)_{\mathbf{p}}$ onto a nontrivial module, so that $(L \otimes_A N)_{\mathbf{p}} \neq 0$ and $\mathbf{p} \in \mathrm{Supp}(L \otimes_A N)$.

Since $\mathbf{p}$ is in the support of both L and N, we have $L_{\mathbf{p}} \neq 0 \neq N_{\mathbf{p}}$. Note that both $L_{\mathbf{p}}$ and $N_{\mathbf{p}}$ are thus finitely generated. Note further that since $A_{\mathbf{p}}$ is local, it's only maximal ideal is $\mathbf{p}A_{\mathbf{p}}$, hence $\mathbf{p}A_{\mathbf{p}}$ is equal to the Jacobson radical of $A_{\mathbf{p}}$ and in particular is contained in it. So by Nakayama's Lemma[3] if $\mathbf{p}L_{\mathbf{p}} = L_{\mathbf{p}}$, then $L_{\mathbf{p}} = 0$. However, $L_{\mathbf{p}} \neq 0$ since $\mathbf{p} \in \mathrm{Supp}(L)$, so we must have that $\mathbf{p}L_{\mathbf{p}} \subset L_{\mathbf{p}}$ is a proper inclusion of submodules. For the same reasons, $\mathbf{p}N_{\mathbf{p}}$ is a proper submodule of $N_{\mathbf{p}}$.

Thus, we have that $L_{\mathbf{p}}/\mathbf{p}L_{\mathbf{p}}$ and $N_{\mathbf{p}}/\mathbf{p}N_{\mathbf{p}}$ are nontrivial vector spaces over the field $k = A_{\mathbf{p}}/\mathbf{p}A_{\mathbf{p}}$. Since these are nontrivial vector spaces, their tensor product $(L_{\mathbf{p}}/\mathbf{p}L_{\mathbf{p}}) \otimes_k$

[2] https://stacks.math.columbia.edu/tag/00L2

[3] In particular, see the second conclusion given in the Stacks project at https://stacks.math.columbia.edu/tag/07RC.

$(N_{\mathbf{p}}/\mathbf{p}N_{\mathbf{p}})$ is nontrivial. Now we will construct a surjective homomorphism

$$(L \otimes_A N)_{\mathbf{p}} \to (L_{\mathbf{p}}/\mathbf{p}L_{\mathbf{p}}) \otimes_k (N_{\mathbf{p}}/\mathbf{p}N_{\mathbf{p}}).$$

We will use the fact that localization of a module is isomorphic as an $A_{\mathbf{p}}$ module to the tensor product with the localization of the ring, by Lemma 10.11.15 of the Stacks Project[4]. In particular, it follows from this lemma, the associativity and commutativity up to isomorphism of the tensor product, and that tensor product with $A_{\mathbf{p}}$ is identity up to isomorphism that

$$(L \otimes_A N)_{\mathbf{p}} \cong L \otimes_A N \otimes_A A_{\mathbf{p}} \cong (L \otimes_A A_{\mathbf{p}}) \otimes_A (N \otimes_A A_{\mathbf{p}}) \cong L_{\mathbf{p}} \otimes_A N_{\mathbf{p}}.$$

So, we will equivalently construct a surjective homomorphism with domain $L_{\mathbf{p}} \otimes_A N_{\mathbf{p}}$. Here, we simply take the tensor product of the projection maps onto the quotients, i.e. we take

$$\pi \otimes \pi : L_{\mathbf{p}} \otimes_A N_{\mathbf{p}} \to (L_{\mathbf{p}}/\mathbf{p}L_{\mathbf{p}}) \otimes_A (N_{\mathbf{p}}/\mathbf{p}N_{\mathbf{p}})$$

given by mapping components of simple tensors to their respective equivalence classes and extending by linearity all to tensors. Hence,

$$\frac{l}{s} \otimes \frac{n}{t} \mapsto \left[\frac{l}{s}\right] \otimes \left[\frac{n}{t}\right]$$

where $[\cdot]$ denotes the respective equivalence classes. This is a tensor of two linear maps, hence is linear, and is clearly surjective. Thus, we have that $L_{\mathbf{p}} \otimes_A N_{\mathbf{p}} \cong (L \otimes_A N)_{\mathbf{p}}$ is nonzero so that $\mathbf{p} \in \mathrm{Supp}(L \otimes_A N)$.

[4] https://stacks.math.columbia.edu/tag/00DK