Commutative Algebra- HW4

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Problem 5.

(a) Let A be a Noetherian ring, and suppose that M is a finitely generated A-module. Let $L, N \leq M$ be sub-modules. Show that $L \subset N$ if and only if $L_{\mathfrak{p}} \subset N_{\mathfrak{p}}$ for all $\mathfrak{p} \in \text{Ass}(M/N)$.

(b) Show that for any subset $U \subset \text{Spec}(A)$, there exists an A-module M such that $U = \text{Ass}(M)$. Show that for any finite $U_0 \subset \text{Spec}(A)$, there exists a finitely generated A-module M_0 such that $U_0 = \text{Ass}(M_0)$.

Definition 1. Let $n \in \mathbb{Z}^+$. Denote $[n] := \{1, 2, \ldots, n\}$.

Theorem 2. Let A be a Noetherian ring, and suppose that M is a finitely generated A-module. Let $L, N \leq M$ be sub-modules. We have that $L \subset N$ if and only if $L_{\mathfrak{p}} \subset N_{\mathfrak{p}}$ for all $\mathfrak{p} \in \text{Ass}(M/N)$.

Proof. Suppose first that $L \subset N$. Let $\mathfrak{p} \in \text{Ass}(M/N)$ be arbitrary. Let $(\ell, s) \in L_{\mathfrak{p}}$. As $L \subset N$, we have that $\ell \in N$. So $(\ell, s) \in N_{\mathfrak{p}}$, and we conclude that $L_{\mathfrak{p}} \subset N_{\mathfrak{p}}$.

Conversely, suppose that $L \not\subset N$. So $(L + N)/N$ is a non-zero A-module. As A is Noetherian, there exists a prime $\mathfrak{p}$ associated to $(L+N)/N$. In particular, $\mathfrak{p} = (N : m)$ for some $m \in (L \setminus N)$. As $(L+N)/N \subset M/N$, it follows that $\text{Ass}((L+N)/N) \subset \text{Ass}(M/N)$. So in particular, $\mathfrak{p} \in \text{Ass}(M/N)$. We claim that $L_{\mathfrak{p}} \not\subset N_{\mathfrak{p}}$. Suppose to the contrary that $L_{\mathfrak{p}} \subset N_{\mathfrak{p}}$. So $(m, 1) \sim (n, s) \in N_{\mathfrak{p}}$, where $n \in N$ and $s \in (A \setminus \mathfrak{p})$. So there exists $u \in (A \setminus \mathfrak{p})$ such that:

$$u(sm - n) = 0.$$

We note that as L is an A-module and $s \in A$, that $sm \in L$. Similarly, $un \in N$. So as $usm = un$, we have that $usm \in N$. Thus, $us \in \mathfrak{p} = (N : m)$. In particular, as $\mathfrak{p}$ is prime, we have that either $u \in \mathfrak{p}$ or $s \in \mathfrak{p}$. This contradicts the assumption that $u, s \notin \mathfrak{p}$. So $L_{\mathfrak{p}} \not\subset N_{\mathfrak{p}}$. The result follows. □

Lemma 3. Let P_1, P_2 be prime ideals. We have that $P_1 \cap P_2$ is prime if and only if $P_1 \subset P_2$ or $P_2 \subset P_1$.

Proof. If $P_1 \subset P_2$ or $P_2 \subset P_1$, then it follows immediately that $P_1 \cap P_2$ is prime. Conversely, suppose that $P_1 \cap P_2$ is prime. As $P_1 \cap P_2$ is an ideal, we have that $P_1P_2 \subset P_1$ and $P_1P_2 \subset P_2$. So $P_1P_2 \subset P_1 \cap P_2$. As $P_1 \cap P_2$ is prime, we have that $P_i \subset P_1 \cap P_2$ for some $i \in [2]$. □

Theorem 4. Let $U \subset \text{Spec}(A)$. There exists an A-module M such that $U = \text{Ass}_A(M)$.

Proof. Define M to be the A-module:

$$M := \bigoplus_{u \in U} A/u.$$

We claim that $U = \text{Ass}_A(M)$. Let $u \in U$. By construction, $u = \text{Ann}_A(A/u)$. As u is prime, $u \in \text{Ass}_A(M)$, and we have that $U \subset \text{Ass}_A(M)$. Conversely, let $\mathfrak{p} \in \text{Ass}_A(M)$, and let $m \in M$ such that $\mathfrak{p}m = 0$. As $m \in M$, we may write:

$$m = \sum_{u \in U} a_u m_u,$$

where each $a_u \in A$ and all but finitely many $a_u \neq 0$. Let $a_1, \ldots, a_k$ denote these non-zero coefficients, and let $u_1, \ldots, u_k \in U$ denote the corresponding primes. It follows that:

$$\mathfrak{p}m = \sum_{i=1}^{k} a_i \mathfrak{p} m_i = 0.$$

So $\mathfrak{p} \subset \mathrm{Ann}_A(m_i) = u_i$ for each $i \in [k]$. Furthermore, if x belongs to each u_i, then $xm_i = 0$. So $x \in \mathfrak{p}$. It follows that:

$$\mathfrak{p} = \bigcap_{i=1}^{k} u_i.$$

However, as $u_1, \ldots, u_k$ are prime, it follows from Lemma 3 that $u_1 = u_2 = \ldots = u_k$. It follows that $\mathfrak{p} \in U$. So $\mathrm{Ass}_A(M) \subset U$. The result follows. $\square$

Remark 5. If U is finite, then:

$$M := \bigoplus_{u \in U} A/u.$$

has only finitely many summands. As A is Noetherian, each summand of M is finitely generated. So M is a finitely-generated A-module.