Commutative Algebra
Assignment 4

Tobias Aldape
Bob Kuo
Connor Meredith

Problem 4. Let M be a finitely generated SI module over a Noetherian ring A.

(a) Show that M is Artinian.

(b) Show that M has a composition series, and that all composition factors are isomorphic.

Lemma 1. *Let M be a finitely generated SI-module, and N be its minimal submodule. Then there exists some $a \in A$ such that the homomorphism $x \mapsto ax$ maps M surjectively onto N.*

Proof. We proceed by induction on the number of generators of M. Suppose $M = \langle m_1 \rangle$ and observe that by minimality, N must be cyclic. Let $N = \langle n \rangle$, then because M is also cyclic, $n = am_1$ for some $a \in A$. Define $f(x) = ax$, then f is clearly a homomorphism. It is surjective because for any $m \in M$, write $m = bm_1$ for some $b \in A$, then $f(m) = abm_1 = bn \in N$.

Now, suppose $M = \langle m_1, \ldots, m_k \rangle$ and suppose that for any SI-module with less than k generators, there exists $a \in A$ such that $x \mapsto ax$ gives a surjective homomorphism into the minimal submodule. Now, N must be the unique minimal submodule of $\langle m_1 \rangle$, so by the induction hypothesis, there exists some $a \in A$ such that $x \mapsto ax$ maps $\langle m_1 \rangle$ onto N surjectively. Define $g(x) = ax$ then $g(M) = \langle g(m_1), \ldots, g(m_k) \rangle$. If each $g(m_i) \in N$, then, $g(M) = N$. So suppose $f(M) \neq N$, then we must have $N \subsetneq g(M)$. Thus, one of $g(m_i)$ for $i > 1$ is in $g(M) \setminus N$ so $N \subsetneq \langle g(m_2), \ldots, g(m_k) \rangle$. Since $g(m_1) \in N \subseteq \langle g(m_2), \ldots, g(m_k) \rangle$, then $g(m_1)$ is a redundant generator for $g(M)$. That is, $g(M) = \langle g(m_2), \ldots, g(m_k) \rangle$ so by the induction hypothesis, there exists $b \in A$ such that $x \mapsto bx$ takes $g(M)$ surjectively onto N. Define $f(x) = bax$ then observe that $f(M) = b \cdot f(M) = N$. □

Proposition 2. *If M is a finitely generated SI module over a Noetherian ring A, then M is Artinian. Furthermore, M has a composition series whose composition factors are all isomorphic.*

Proof. Let N be the minimal submodule of M. If $M = N$, then the theorem is trivially true. So suppose $M \neq N$. Applying Lemma 1, there exists a surjective homomorphism $f_1 : M \to N$ defined by $f_1(x) = a_1 x$ for $a_1 \in A$. Denote $M = M_0$ and $M_1 = \ker f_1$. Since f_1 is surjective, then $N \cong M/\ker f_1 = M_0/M_1$. So long as $N \subseteq M_{i-1}$, we may continue applying Lemma 1 to define surjective homomorphisms $f_i : M_{i-1} \to N$ given by $f_i(x) = a_i x$ for $a_i \in A$ and defining $M_i = \ker f_i$. This gives a descending chain

$$M = M_0 \geq M_1 \geq M_2 \geq \cdots$$

of submodules of A such that $M_{i-1}/M_i \cong N$. Suppose that there is some n such that $M_n = N$, then $f_{n+1} : M_n \to N$ is an isomorphism, and $M_{n+1} = \ker f_{n+1} = 0$. This gives the composition series

$$M = M_0 \geq M_1 \geq M_2 \geq \cdots \geq M_n = N \geq M_{n+1} = 0,$$

whose composition factors are all isomorphic to N.

On the other hand, if there does not exist any i such that $M_i = N$, then this chain of submodules can be extended indefinitely. Let $I_i = \mathrm{Ann} M_i$ and observe that because

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$M_i \leq M_{i-1}$ then $I_i \geq I_{i-1}$. Furthermore, $a_i \in I_i$ because $M_i = \ker f_i$ and $f_i(x) = a_i x$. However, $f_i(M_{i-1}) = N$, so $a_i \notin I_{i-1}$. Thus, we obtain a strictly ascending chain of ideals

$$I_1 < I_2 < I_3 < \cdots$$

that continue indefinitely. Since A is Noetherian, then it cannot have such a chain. Thus, $M_0 \geq M_1 \geq M_2 \geq \cdots$ must terminate to the composition series described above.

Note that this gives a finite chain of intervals from 0 to M each satisfying DCC (because each interval is simple). So by modularity, the entire lattice of submodules of M satisfies DCC. $\square$