COMMUTATIVE ALGEBRA HOMEWORK 4

CHASE MEADORS, ADRIAN NEFF, HOWIE JORDAN

Problem (3)**.** A ring is *subdirectly irreducible* (SI) if it has a *least* nonzero ideal. Prove in each of the following ways that an SI Noetherian ring must be Artitian.

(a) Using primary decomposition: Show that A has a unique associated prime, which is a nilpotent maximal ideal. Then show that a Noetherian ring with a nilpotent maximal ideal is Artinian.

(b) Using the Krull Intersection Theorem: again, first show that A has a nilpotent maximal ideal.

Claim. An SI Noetherian ring A has a unique associated prime, which is a maximal nilpotent ideal.

Proof. (Via primary decomposition). Any intersection of nonzero ideals (particularly primary ideals) must lie above I (since all nonzero ideals contain I). Associated primes of A come from primary decompositions of (0), but any intersection yielding (0) must be trivial; in particular, it must contain the zero ideal, thus (0) is a primary ideal and the only irredundant primary decomposition of (0) is (0). This means the only associated prime of A is the nilradical $\sqrt{(0)} = \mathfrak{N}$, which also must be the set of zero divisors of A. By calg1p6, $(0 : I)$ is a maximal ideal, and since it consists of zero divisors, we have $(0 : I) \leq \mathfrak{N}$ and thus $(0 : I) = \mathfrak{N}$. Thus $\mathfrak{N}$ is maximal and, since A is Noetherian, nilpotent. $\square$

Proof. (Via Krull intersection theorem). Since I is minimal, $\mathfrak{m} = (0 : I)$ is maximal, by calg1p6. Suppose toward contradiction that $\mathfrak{m}$ is not nilpotent. That is, every power of $\mathfrak{m}$ is non-zero. Then every power of $\mathfrak{m}$ contains I, and thus $K = \bigcap_{i=1}^{\infty} \mathfrak{m}^i$ contains I. By the Krull intersection theorem, there is an $m \in \mathfrak{m}$ so that $1 - m$ annihilates K. But then, $1 - m$ annihilates I; this is impossible since that would imply $I = mI$, but m itself annihilates I. Thus, $\mathfrak{m}$ is actually nilpotent. $\square$

Claim. A Noetherian ring with a nilpotent maximal ideal is Artinian.

Proof. Let A be Noetherian with nilpotent maximal ideal $\mathfrak{m}$. Then consider the finite chain $A > \mathfrak{m} \geq \mathfrak{m}^2 \geq \cdots \geq \mathfrak{m}^n = (0)$. Each $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is an A-module and moreover, since $\mathfrak{m}(\mathfrak{m}^i) = \mathfrak{m}^{i+1}$, an $A/\mathfrak{m}$-module, which is a vector space. Vector spaces have complemented modular subspace lattices, thus the subspace lattice of each factor (which is the same of the interval $[\mathfrak{m}^{i+1}, \mathfrak{m}^i]$ in the ideal lattice of A) has DCC since it has ACC. Then we have a finite collection of intervals with DCC spanning the ideal lattice of A; since the ideal lattice is modular, we conclude that the whole ideal lattice of A has DCC. $\square$