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2. Let M be an A-module, and let $m, n \in M$ and $r \in A$ be elements.

(a) Show that the set of primes $\mathfrak{p}$ where $m = 0$ is an open subset of $\mathrm{Spec}(A)$, and that it is all of $\mathrm{Spec}(A)$ iff $m = 0$. (Here $m \in M_{\mathfrak{p}}$ is shorthand for $m/1 \in M_{\mathfrak{p}}$)

(b) Show that the set of primes $\mathfrak{p}$ where $m = n$ is an open subset of $\mathrm{Spec}(A)$, and that it is all of $\mathrm{Spec}(A)$ iff $m = n$.

(c) Show that the set of primes $\mathfrak{p}$ where r is nilpotent is an open subset of $\mathrm{Spec}(A)$, and that it is all of $\mathrm{Spec}(A)$ iff r is nilpotent.

(d) Show that the set of primes $\mathfrak{p}$ where r is a unit is an open subset of $\mathrm{Spec}(A)$, and that it is all of $\mathrm{Spec}(A)$ iff r is a unit.

Proof.

A note regarding notation: superscript c (e.g $\mathfrak{p}^c$) denotes the set complement, NOT the contraction of an ideal.

(a) $m = 0 \iff m$ is in the 0 submodule. We have a lemma (*Primary decomposition, slide 15*) which states that for a submodule N, the set of primes where $m \in N_{\mathfrak{p}}$ is open in $\mathrm{Spec}(A)$, hence the set of primes where $m/1$ is open.

Moreover, the lemma states that this set of primes is all of $\mathrm{Spec}(A)$ iff $m \in N$, which in this case means iff $m = 0$.

(b) $m = n$ in $M_{\mathfrak{p}}$ iff $\exists u \in \mathfrak{p}^c$ s.t $u(m-n) = 0$ i.e $u \in (0 : m-n)$. This holds iff $(0 : m-n) \not\subseteq \mathfrak{p}$ iff $\mathfrak{p} \in V((0 : m-n))^c$ which is open.

$m - n = 0$ in all prime localizations iff $m - n = 0$ in M by part (a).

(c) Consider the sets $\alpha_1 = \{\mathfrak{p} | r = 0 \text{ in } A_{\mathfrak{p}}\}$, $\alpha_2 = \{\mathfrak{p} | r^2 = 0 \text{ in } A_{\mathfrak{p}}\}$, $\alpha_3 = \{\mathfrak{p} | r^3 = 0 \text{ in } A_{\mathfrak{p}}\}$... By part (a) each α_i is an open set in $\mathrm{Spec}(A)$. The union $\bigcup_{i>0} \alpha_i$ is the set of primes where r is nilpotent and it's open in $\mathrm{Spec}(A)$ since it's the union of open sets.

If r is nilpotent, then it's nilpotent in every localization. Conversely, if $r^n = 0$ in $A_{\mathfrak{p}}$ then $\exists u \in \mathfrak{p}^c$ s.t $ur^n = 0$ and since $0 \in \mathfrak{p}$ and $u \notin \mathfrak{p}$ this means that $r^n \in \mathfrak{p} \implies r \in \mathfrak{p}$. Therefore, if r becomes nilpotent for every prime, then it's contained in every prime, and hence it's in $\mathfrak{N}$.

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(d) r is a unit in $A_{\mathfrak{p}}$ iff $\exists$ an element a/s such that $ra/s = 1$. In other words, $\exists u, s \in \mathfrak{p}^c$ s.t $u(ar - s) = 0$. This holds iff $(r) \cap \mathfrak{p}^c \neq \emptyset$ iff $(r) \not\subseteq \mathfrak{p}$. Therefore, the set of primes for which r is a unit is the complement of the closed set $V((r))$.

If r is a unit then it's a unit in every localization. Conversely, if $(r) \not\subseteq \mathfrak{p}$ for all primes then r is in no maximal ideal, therefore it's a unit. $\square$