# Commutative Algebra HW4p1

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Let $L, N \leq M$ be $A$-modules. Let $U$ be the set of primes for which $L_p \subseteq N_p$ holds. Show that $U$ is an intersection of open sets in $\operatorname{Spec}(A)$. Show conversely that if $V$ is any intersection of open sets in $\operatorname{Spec}(A)$ then $V$ is exactly the set of primes for which $L_p \subseteq N_p$ holds for some submodules $L, N$ of some module $M$.

*Proof.* First, suppose that $L$ and $N$ are submodules of the module $M$. Then suppose $q \subseteq p$ are primes of $A$ and $L_p \subseteq N_p$. Then

$$L_q = (p \backslash q)^{-1}(L_p) \subseteq (p \backslash q)^{-1} N_p = N_q.$$

It follows that the collection $U$ for which the inequality holds is downward closed when ordered by inclusion. Then the collection of primes $\operatorname{Spec}(A) \backslash U$ for which the inequality fails must be upward closed. Therefore

$$\operatorname{Spec}(A) \backslash U = \bigcup_{p \in \operatorname{Spec}(A) \backslash U} V(p),$$

showing that $\operatorname{Spec}(A) \backslash U$ is a union of closed sets. Therefore its complement $U$ is the intersection of open sets.

First we prove the converse direction when $V$ is some open set. So let $V \subseteq \operatorname{Spec}(A)$ be open. Then $\operatorname{Spec}(A) \backslash V$ is closed. There must be some ideal $I \trianglelefteq A$ such that $\operatorname{Spec}(A) \backslash V = V(I)$ Let $L, M = A/I$ and let $N = 0$, the zero submodule. These are all $A$-modules/submodules. Then $(A/I)_p = L_p \subseteq N_p = 0$ if and only if for all $a + I \in A/I$ there is some $t \in A \backslash p$ such that $t(a + I) = 0 + I$.

If this holds, then given $1 + I \in A/I$ there must be $t \in A \backslash p$ such that $t + I = t(1 + I) = 0 + I$, so that $t \in I$. In other words, we must have $(A \backslash p) \cap I \neq \varnothing$, which is equivalent to $I \not\subseteq p$. Conversely, suppose that $I \not\subseteq p$. Then there must be $t \in I$ such that $t \notin p$. Then given $a + I \in A/I$, we know that $ta \in I$ and $t(a + I) = ta + I = 0 + I$. This establishes that $(A/I)_p \subseteq 0$. Therefore $L_p \subseteq N_p$ if and only if $I \not\subseteq p$, which happens if $p \notin V(I)$, which happens if and only if $p \in U$.

Now suppose that $V = \bigcap_{j \in J} V_j$ is the intersection of some collection of open sets. Let $I_j \trianglelefteq A$ be such that $V(I_j) = \operatorname{Spec}(A) \backslash U_j$. Let $L, M = \bigoplus_{j \in J} A/I_j$ and let $N$ be the zero submodule. We have already shown that $(A/I_j)_p \subseteq 0$ if and only if $p \in V_j$. Then

$$\left( \bigoplus_{j \in J} (A/I_j) \right)_p = \bigoplus_{j \in J} (A/I_j)_p \subseteq 0$$

if and only if $(A/I_j)_p \subseteq 0$ for all $j \in J$, which happens if and only if $p \in V_j$ for all $j \in J$, which happens if and only if $p \in \bigcap_{j \in J} V_j$. $\square$