Toby Aldape
Michael Levet
Adrian Neff

9. Show that the class $\mathcal{P}$ of projective R-modules is closed under $\oplus$ and $\otimes$ and contains 0 and R. Show that the same is true if we replace $\mathcal{P}$ with the subclass $\mathcal{P}_{\text{f.g.}}$ of finitely generated projective R-modules.

Proof: We use the fact that an R-module is projective if and only if it is a direct summand of a free module.

First, R is free, so it is projective, and $R \cong R \oplus 0$ exhibits 0 as a direct summand of a free module, so 0 is projective. Next, suppose that M and N are projective R-modules, so there exist R-modules E, F, Q, and P, with E and F free, such that

$$E \cong Q \oplus M$$

$$F \cong P \oplus N.$$

A direct sum of free R-modules is free, so $E \oplus F$ is free. Therefore,

$$E \oplus F \cong Q \oplus P \oplus (M \oplus N)$$

exhibits $M \oplus N$ as a direct summand of a free R-module, so $M \oplus N$ is projective. A tensor product of free R-modules is also free, as tensor products distribute over direct sums and $R \otimes R \cong R$, so $E \otimes F$ is free. Therefore,

$$E \otimes F \cong (Q \otimes P) \oplus (Q \otimes N) \oplus (M \otimes P) \oplus (M \otimes N)$$

exhibits $M \otimes N$ as a direct summand of a free R-module, so $M \otimes N$ is projective. Hence $\mathcal{P}$ is closed under $\oplus$ and $\otimes$ and contains 0 and R.

For the second statement, we need only show that 0 and R are finitely generated and that direct sums and tensor products of finitely generated R-modules are finitely generated, as the rest of the statement follows from the previous statement. We know that 0 is generated by 0 and R is generated by 1, so 0 and R are finitely generated. Let M and N be finitely generated R-modules, say M is generated by $\{m_1, ..., m_r\}$ and N is generated by $\{n_1, ..., n_s\}$. We see that $M \oplus N$ is generated by $\{(m_i, 0) : 1 \leq i \leq r\} \cup \{(0, n_j) : 1 \leq j \leq s\}$ and $M \otimes N$ is generated by $\{m_i \otimes n_j : 1 \leq i \leq r, 1 \leq j \leq s\}$, so $M \oplus N$ and $M \otimes N$ are finitely generated R-modules. Hence $\mathcal{P}_{\text{f.g.}}$ is closed under $\oplus$ and $\otimes$ and contains 0 and R.