COMMUTATIVE ALGEBRA HOMEWORK 3

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Problem (8)**.** Show that if A is a flat R-module, then its character module $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ is an injective R-module (the converse is also true, but you don't have to prove it).

We start with a useful but not quite immediate characterization of injective modules:

Lemma 1. *Q is an injective R-module if and only if for any injection $i : S \hookrightarrow M$ and map $f : S \to Q$, f can be extended to a map $f' : M \to Q$ with $f = f' \circ i$:*

$$\begin{tikzcd} & Q & \\ 0 \arrow[r, hook] & S \arrow[u, "f"] \arrow[r, hook, "i"'] & M \arrow[ul, dotted, "\exists f'"'] \end{tikzcd}$$

Proof. Given a SES starting with $0 \to S \overset{i}{\hookrightarrow} M$, we need only check exactness at the right end of the image under $\mathrm{Hom}(-, Q)$, which is $\mathrm{Hom}(M, Q) \xrightarrow{-\circ i} \mathrm{Hom}(S, Q) \to 0$; that is, that $- \circ i$ is surjective. This precisely means that every map $S \to Q$ can be extended through the injection i to a map $M \to Q$. $\square$

Corollary. Q is an injective R-module if and only if any map $I \to Q$ from an ideal I of R extends to a map $R \to Q$.

Proof. By Baer's criterion, we need only check that $\mathrm{Hom}(-, Q)$ is exact on a SES of the form $0 \to I \to R \to R/I \to 0$; apply the previous lemma to such a sequence. $\square$

Lemma 2. *$\mathbb{Q}/\mathbb{Z}$ is a injective $\mathbb{Z}$-module.*

Proof. By the last two results, we only need to check that a map $f : n\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ extends to a map $f' : \mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$. Indeed, define f' by taking 1 to $f(n)/n$. $\square$

Claim (Problem 8)**.** If A is a flat R-module, then its character module is injective.

Proof. By lemma 1, we need to ensure that the following diagram can be completed for any injection $i : S \hookrightarrow M$ and $f : S \to \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$:

$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr, hook, "i"'] & & \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}) \\ & M \arrow[ur, dotted] & \end{tikzcd}$$

The most general form of the tensor hom adjunction gives a natural isomorphism:

$$(*) \qquad \mathrm{Hom}_{\mathbb{Z}}({}_{\mathbb{Z}}A_R \otimes_R {}_R X_{\mathbb{Z}}, {}_{\mathbb{Z}}\mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_R({}_R X_{\mathbb{Z}}, \mathrm{Hom}_{\mathbb{Z}}({}_{\mathbb{Z}}A_R, {}_{\mathbb{Z}}\mathbb{Q}/\mathbb{Z}))$$

Every module has a $\mathbb{Z}$-module on either side, so S and M have the appropriate module structures to play the role of X in this isomorphism. The map f lives on the right-hand side of $(*)$. Thus, applying this natural isomorphism to the whole diagram yields:

$$\begin{tikzcd} A \otimes_R S \arrow[rr, "\overline{f}"] \arrow[dr, hook, "A\otimes_R i"'] & & \mathbb{Q}/\mathbb{Z} \\ & A \otimes_R M \arrow[ur, dotted, "g"'] & \end{tikzcd}$$

where $\overline{f}$ denotes the adjunct of f under the the natural isomorphism; both $\overline{f}$ and the yet-undefined map g live on the left side of $(*)$ while the left map is the image of i under $A \otimes_R -$. We are assured that the left map is an injection since A is flat (hence $A \otimes_R -$ is exact and maps injections to injections). By 2, $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module. Thus, we may lift the map $\overline{f}$ to the indicated map $g \in \mathrm{Hom}_{\mathbb{Z}}(A \otimes_R M, \mathbb{Q}/\mathbb{Z})$ that completes the triangle. Then the adjunct $\overline{g} \in \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}))$ of g completes the original triangle: indeed, naturality guarantees that $f = \overline{g \circ i \otimes_{\mathbb{Z}} A} = \overline{g} \circ i$. $\square$