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We say a module M is finitely generated if there exists surjective homomorphism $\oplus^k R \to M$ for some positive integer k.

Lemma 0.1. *Suppose that $0 \to A \to B \to C \to 0$ is an exact sequence of modules. Assume A and C are finitely generated so that you have the diagram below.*

$$\begin{tikzcd}
 & \oplus^m R \arrow[d, "\pi_A"'] & & \oplus^n R \arrow[d, "\pi_C"'] & \\
0 \arrow[r] & A \arrow[r, "\gamma"] & B \arrow[r, "\delta"] & C \arrow[r] & 0
\end{tikzcd}$$

Then B is finitely generated. In particular, if there are surjections $\oplus^m R \to A$ and $\oplus^n R \to C$, then there exists a surjection $(\oplus^m R) \oplus (\oplus^n R) \to B$ such that the following diagram commutes and the two rows are exact.

$$\begin{tikzcd}
0 \arrow[r] & \oplus^m R \arrow[r, "\mu"] \arrow[d, "\pi_A"'] & (\oplus^m R) \oplus (\oplus^n R) \arrow[r, "\eta"] \arrow[d, "\pi_B"'] & \oplus^n R \arrow[r] \arrow[d, "\pi_C"'] & 0 \\
0 \arrow[r] & A \arrow[r, "\gamma"] & B \arrow[r, "\delta"] & C \arrow[r] & 0
\end{tikzcd}$$

where μ is inclusion in the first coordinate and η is projection of the second coordinate.

Proof. Assume A and C are both finitely generated. Then we have surjective homomorphisms $\oplus^m R \to A$ and $\oplus^n R \to C$. Then we have most of the diagram in the lemma, only missing $(\oplus^m R) \oplus (\oplus^n R)$ and the arrows coming in and out of it.

We have $\gamma \circ \pi_A : \oplus^m R \to B$. Also, $\oplus^n R$ is a free module and in particular a projective module. By exactness, we have that δ is a surjective map. Then by projective lifting property we have a homomorphism $\tilde{\pi_C} : \oplus^n R \to B$. These two maps, $\gamma \circ \pi$ and $\tilde{\pi_C}$ induce a map from the coproduct $\pi_B : (\oplus^m R) \oplus (\oplus^n R) \to B$ defined by $(x, y) \mapsto \gamma \circ \pi_A(x) + \tilde{\pi_C}(y)$. We must show surjectivity next.

Take $b \in B$. $\delta(b) \in C$ so by surjectivity of π_C there must exist some $y \in \oplus^n R$ such that $\pi_C(y) = \delta(b)$. By construction, $\delta \circ \tilde{\pi_C}(y) = \pi_C(y)$. So $b - \tilde{\pi_C}(y) \in \ker(\delta)$. By exactness, $b - \tilde{\pi_C}(y) \in \operatorname{Im}(\gamma)$. The map π_A is also surjective, so we have $b - \tilde{\pi_C}(y) \in \operatorname{Im}(\gamma \circ \pi_A)$. Written out, for some $x \in \oplus^m R$, $b - \tilde{\pi_C}(y) = \gamma \circ \pi_A(x)$, or equivalently $b = \gamma \circ \pi_A(x) + \tilde{\pi_C}(y) = \pi_B(x, y)$.

Now to show that the top row is exact, (the bottom one is exact by assumption). η is surjective since for any $y \in \oplus^n R$, we have $\eta(0, y) = y$. So $\ker(0) = \operatorname{Im}(\eta) = \oplus^n R$. We also have that $(x, y) \in \ker(\eta)$, then $y = 0$ and $\mu(x) = (x, y)$. If $(x, y) \in \operatorname{Im}(\mu)$, then $y = 0$ and $\eta(x, y) = y = 0$. So we have that $\operatorname{Im}(\mu) = \ker(\eta)$. Lastly, μ is injective because if $\mu(x) = \mu(y)$ then $(x, 0) = (y, 0)$ which only happens if $x = y$. So $\operatorname{Im}(0) = \ker(\mu) = 0$.

Now we show that the diagram commutes. Let $x \in \oplus^m R$. We know that $\pi_B(\mu(x)) = \pi_B(x, 0) = \gamma \circ \pi_A(x) + 0 = \gamma \circ \pi_A(x)$, so the first square is commutative. Now consider $(x, y) \in (\oplus^m R) \oplus (\oplus^n R)$. We know that $\delta(\pi_B(x, y)) = \delta(\gamma(\pi_A(x)) + \tilde{\pi}_C(y)) = \delta(\gamma(\pi_A(x))) + \delta(\tilde{\pi}_C(y)) = \delta(\tilde{\pi}_C(y))$ since the image of γ is contained in the kernel of δ. Also, $\pi_C(\eta(x, y)) = \pi_C(y) = \delta(\tilde{\pi}_C(y))$, so the second square is commutative.

□

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7. An R-module M is finitely presentable iff there is an exact sequence

$$\oplus^n R \xrightarrow{\alpha} \oplus^m R \xrightarrow{\beta} M \to 0.$$

Show that if $0 \to A \to B \to C \to 0$ is an exact sequence of modules and if A and C are finitely presentable, then so is B.

Proof. Assume A and C are finitely presentable. Then we have the following diagram.

$$\begin{tikzcd}
 & \oplus^p R \arrow[d, "\alpha_A"'] & & \oplus^q R \arrow[d, "\alpha_C"'] & \\
 & \oplus^m R \arrow[d, "\beta_A"'] & & \oplus^n R \arrow[d, "\beta_C"'] & \\
0 \arrow[r] & A \arrow[r, "\gamma"] \arrow[d] & B \arrow[r, "\delta"] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 &
\end{tikzcd}$$

By the lemma, we have that there exists a surjective $\beta_B : (\oplus^m R) \oplus (\oplus^n R) \to B$. It is enough to show that this map has finitely generated kernel, for then we can construct a surjective map $\oplus^k R \to \ker(\beta_B)$.

The Snake Lemma can be applied as can be seen by the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & \oplus^m R \arrow[r] \arrow[d] & (\oplus^m R) \oplus (\oplus^n R) \arrow[r] \arrow[d] & \oplus^n R \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

We then have an exact sequence $0 \to \ker \beta_A \to \ker \beta_B \to \ker \beta_C \to 0$. β_A is surjective and that is why we have a 0 in place of its cokernel. $\ker \beta_A \to \ker \beta_B$ is injective because the coprojection map $\oplus^m R \to (\oplus^m R) \oplus (\oplus^n R)$ is injective.

Since the image of α_X equals the kernel of β_X for $X \in \{A, C\}$, we obtain the following diagram, where the vertical arrows are surjective.

$$\begin{tikzcd}
 & \oplus^p R \arrow[d, "\alpha_A"'] & & \oplus^q R \arrow[d, "\alpha_B"'] & \\
0 \arrow[r] & \ker \beta_A \arrow[r] & \ker \beta_B \arrow[r] & \ker \beta_C \arrow[r] & 0
\end{tikzcd}$$

This is the assumption of our lemma, so we obtain a surjection from $(\oplus^p R) \oplus (\oplus^q R)$ onto $\ker \beta_B$. That is we have a map $\alpha_B : (\oplus^p R) \oplus (\oplus^q R) \to (\oplus^m R) \oplus (\oplus^n R)$ and $\operatorname{Im} \alpha_B = \ker \beta_B$. We then have the exact sequence $(\oplus^p R) \oplus (\oplus^q R) \to (\oplus^m R) \oplus (\oplus^n R) \to B \to 0$. We have that B is finitely presentable.

$\square$