Homework 3 Problem 6

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Suppose that

$$0 \to \mathrm{Hom}(C, M) \xrightarrow{\circ\psi} \mathrm{Hom}(B, M) \xrightarrow{\circ\phi} \mathrm{Hom}(A, M) \to 0$$

is exact for every R-module M. Show that

$$A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$$

is exact. (Hint: Consider what happens when $M = C/\mathrm{im}\ \psi$, $M = C$ and $M = B/\mathrm{im}\ \phi$.)

Proof. Let $M = C/\mathrm{im}\ \psi$. Then the sequence

$$0 \to \mathrm{Hom}(C, C/\mathrm{im}\ \psi) \xrightarrow{\circ\psi} \mathrm{Hom}(B, C/\mathrm{im}\ \psi)$$

is exact and the $\circ\psi$ map is injective. Let π_ψ be the quotient map $\pi : C \to C/\mathrm{im}\ \psi$. Then $0, \pi_\psi \in \mathrm{Hom}(C, C/\mathrm{im}\ \psi)$. But $0 \circ \psi = 0 = \pi_\psi \circ \psi$. The injectivity of the $\circ\psi$ map shows $0 = \pi_\psi$. Since $\pi_\psi : C \to C/\mathrm{im}\ \psi$ is a quotient map, this shows that $\mathrm{im}\ \psi = C$. Therefore ψ is surjective.

Now let $M = C$. We have that the sequence

$$\mathrm{Hom}(C, C) \xrightarrow{\circ\psi} \mathrm{Hom}(B, C) \xrightarrow{\circ\phi} \mathrm{Hom}(A, C)$$

is exact. Since $\mathrm{Id}_C \in \mathrm{Hom}(C, C)$, it follows that $0 = \mathrm{Id}_C \circ \psi \circ \phi = \psi \circ \phi$. Therefore $\mathrm{im}\ \phi \subseteq \ker\ \psi$.

Now, it suffices to show that $\ker\ \psi \subseteq \mathrm{im}\ \phi$. To do this, let $M = B/\mathrm{im}\ \phi$. The sequence

$$\mathrm{Hom}(C, B/\mathrm{im}\ \phi) \xrightarrow{\circ\psi} \mathrm{Hom}(B, B/\mathrm{im}\ \phi) \xrightarrow{\circ\phi} \mathrm{Hom}(A, B/\mathrm{im}\ \phi)$$

is exact. Let $\pi_\phi : B \to B/\mathrm{im}\ \phi$ be the quotient map. Then $\pi_\phi \in \mathrm{Hom}(B, B/\mathrm{im}\ \phi)$ and $\pi_\phi \circ \phi = 0$, therefore π_ϕ is in the kernel of the $\circ\phi$ map and thus is in the image of the $\circ\psi$ map. Then, for some $\chi \in \mathrm{Hom}(C, B/\mathrm{im}\ \phi)$, we have $\chi \circ \psi = \pi_\phi$, so $\ker\ \psi \subseteq \ker\ \pi_\phi = \mathrm{im}\ \phi$. □