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5. Let $\phi : R \to S$ be a ring homomorphism. Show that the functor of extension of scalars from R-Mod to S-Mod induces a commutative semi-ring homomorphism from $\langle R\text{-Mod}; \otimes, \oplus, 0, R\rangle \cong\to \langle S\text{-Mod}; \otimes, \oplus, 0, S\rangle / \cong$.

Proof. We need to prove that the map induced by the extension of scalars functor, $E : R\text{-Mod} \to S\text{-Mod}$, respects all the semi-ring data. Since E is a functor, it sends isomorphisms to isomorphisms, so, it's sufficient to work with class representatives and show that the data is preserved up to isomorphism.

As a matter of notation, $S \otimes_R A$ denotes the tensor product in R (where S is given the restriction of scalars structure), meanwhile, $\overline{S \otimes_R A}$ is the S-Module with the action $s' \cdot (s \otimes a) = s's \otimes a$. The construction $S \otimes_R A \mapsto \overline{S \otimes_R A}$ is a functor from the subcategory $\mathrm{Im}(S \otimes_R \_) \leq S$-Mod to S-Mod.

The Multiplicative Unit:

$E(A) = \overline{S \otimes_R A}$, therefore, $E(R) = \overline{S \otimes_R R}$ and since $S \otimes_R R \cong S$, we have $\overline{S \otimes_R R} \cong \overline{S}$. But S acts on $\overline{S}$ by ring multiplication, therefore, $\overline{S} \cong S$.

Multiplication:

$\overline{S \otimes_R (A \otimes_R B)}$ is S-isomorphic to $\overline{(S \otimes_R (A \otimes_R B))} \otimes_S S$, so we need an S-isomorphism between this module and $\overline{(S \otimes_R A)} \otimes_S \overline{(S \otimes_R B)}$. Define ϕ on simple tensors $(s \otimes (a \otimes b)) \otimes s' \mapsto (s \otimes a) \otimes (s' \otimes b)$ and extend by linearity. With the obvious inverse, ϕ is an S-linear isomorphism.

Addition:

$S \otimes_R (A \oplus B) \cong (S \otimes_R A) \oplus (S \otimes_R B)$ as R-modules. This isomorphism is given explicitly by $\phi : s \otimes (a, b) \mapsto (s \otimes a, s \otimes b)$. We can check the compatibility of the $\mathbb{Z}$-Mod isomoprhism ϕ with the S action:

$$\phi(s' \cdot (s \otimes (a, b))) = \phi((s's) \otimes (a, b)) = (s's \otimes a, s's \otimes b) = s'(s \otimes a, s \otimes b) = s'\phi(s \otimes (a, b))$$

So if we give $(S \otimes_R A) \oplus (S \otimes_R B)$ the S-structure $\overline{(S \otimes_R A)} \oplus \overline{(S \otimes_R B)}$, we have that the $\mathbb{Z}$-isomorphism ϕ is also an S-isomorphism $\overline{S \otimes_R (A \oplus B)} \to \overline{(S \otimes_R A)} \oplus \overline{(S \otimes_R B)}$

The Additive Identity:

$E(0) = \overline{S \otimes_R 0}$ but $S \otimes_R 0 \cong 0$ and $\overline{0}$ is the 0 module in S-Mod.

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