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4. (There is no contravariant analogue of the tensor product)

(a) Let k-Vec denote the category of vector spaces over the field k. Show that the double dual functor $V \mapsto V^{**}$ is an additive covariant functor that is not representable.

(b) A contravariant version of the tensor product, say $B \boxtimes_R C$, might be expected to satisfy the property that it represents the composite of the contravariant representable functors $\mathrm{Hom}_R(-, B)$ and $\mathrm{Hom}_R(-, C)$. Show that there is no such general construction for categories of modules.

Proof.

(a) First we will show that the assignment $(-)^{**} : k\text{-Vec} \to k\text{-Vec}$ is indeed a covariant functor. Let $f : V \to W$ be a linear map. Note that $f^{**} : V^{**} \to W^{**}$ must take some $\Phi \in V^{**}$, hence a $\Phi : V^* \to k$, and produce an element $f^{**}\Phi \in W^{**}$, that is $f^{**}\Phi : W^* \to k$. Note that if $w \in W^*$, that is if $w : W \to k$ is a linear map, then $w \circ f$ is a linear map $V \to k$, hence $w \circ f \in V^*$. So, we can define

$$(f^{**}\Phi)(w) = \Phi(w \circ f).$$

To see that this covariant assignment is a functor, we must show preservation of identity morphisms and composition. For the identity morphism $\mathrm{id}_V : V \to V$ for some vector space V, the map $\mathrm{id}_V^{**} : V^{**} \to V^{**}$ acts by mapping a $\Phi : V^* \to k$ to the map $(\mathrm{id}_V^{**}\,\Phi) : V^* \to k$. By the definition given above and the fact that $w \circ \mathrm{id}_V = w$ for all $w : V \to k$,

$$(\mathrm{id}_V^{**}\,\Phi)(w) = \Phi(w \circ \mathrm{id}_V) = \Phi(w)$$

so that $\mathrm{id}_V^{**} = \mathrm{id}_{V^{**}}$. To show composition, let $g : U \to V$ be another linear map for some $U \in k$-Vec. We must show that $(f \circ g)^{**} = f^{**} \circ g^{**}$. Let $\Phi \in U^{**}$, $w \in W^*$ Then

$$((f \circ g)^{**}\Phi)(w) = \Phi(w \circ (f \circ g)).$$

As $w \circ (f \circ g) = (w \circ f) \circ g$, we have then that

$$\begin{aligned}
\Phi(w \circ (f \circ g)) &= \Phi((w \circ f) \circ g) \\
&= (g^{**}\Phi)(w \circ f) \\
&= (f^{**}(g^{**}\Phi))(w) \\
&= ((f^{**} \circ g^{**})\Phi)(w).
\end{aligned}$$

Thus, we have that $(f \circ g)^{**} = f^{**} \circ g^{**}$ so that the double dual is indeed a functor.

To see that this functor is additive, we must show that it preserves finite biproducts. That it preserves the zero object 0 (i.e., the nullary biproduct) follows from observing that $0^* = \mathrm{Hom}(0, k) \cong 0$ as there is only the one unique linear map $0 \to k$, hence $0^{**} = (0^*)^* \cong 0^* \cong 0$. For a binary biproduct $V \oplus W$, consider

$$(V \oplus W)^{**} = \mathrm{Hom}(\mathrm{Hom}(V \oplus W, k), k).$$

Since biproduct is in particular a coproduct, we have then that this naturally isomorphic to

$$\mathrm{Hom}(\mathrm{Hom}(V,k)\times\mathrm{Hom}(W,k),k).$$

But then, the product $\times$ is again actually the biproduct $\oplus$, hence is also a coproduct, so we have a natural isomorphism with

$$\mathrm{Hom}(\mathrm{Hom}(V,k),k)\times\mathrm{Hom}(\mathrm{Hom}(W,k),k)\cong V^{**}\oplus W^{**}.$$

So, the double dual functor is additive.

However, the double dual functor is not representable. If there were some representing object, say some vector space A such that $V^{**}\cong\mathrm{Hom}(A,V)$ for all $V\in k$-Vec, then we must have the dimensions are equal, i.e. that $\dim(V^{**})=\dim(\mathrm{Hom}(A,V))$ for all V. But this cannot generally be the case.

To see this, recall that if V is finite dimensional, then $V^*\cong V$ and we have $\dim(V^{**})=\dim(V)$. This implies that the representing object A should have dimension 1, so that $\dim(\mathrm{Hom}(A,V))=\dim(A)\dim(V)=\dim(V^{**})=\dim(V)$. However, when the dimension of V is infinite, $\dim(V^*)$ is strictly greater than $\dim(V)$, implying that the representing object should have dimension higher than 1, a contradiction. Hence, no such representing object for the double dual functor can exist.

(b) Consider the case where $R=k$ a field and $B=C=k$ as a k vector space. Then $\mathrm{Hom}_k(\mathrm{Hom}_k(-,k),k)=(-)^{**}$. Hence, if this composite functor were representable we would have a representing object for the double dual, and by part (a) the double dual functor is not representable. Hence, there can be no analogous contravariant tensor product for k-Vec and hence no such construction for categories of R modules in general.