Commutative Algebra- HW3

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Problem 3. Let m be an integer that is not a perfect square.

(a) Show that $\mathbb{Q}[\sqrt{m}] \otimes_{\mathbb{Q}} \mathbb{Q}[\sqrt{m}] \cong \mathbb{Q}[\sqrt{m}] \times \mathbb{Q}[\sqrt{m}]$ as $\mathbb{Q}$-algebras.

(b) Find the idempotents in $\mathbb{Q}[\sqrt{m}] \otimes_{\mathbb{Q}} \mathbb{Q}[\sqrt{m}]$ that induce the direct decomposition in (a).

(c) Find an idempotent $e \neq 0, 1$ in $\mathbb{Q}[\sqrt[3]{2}] \otimes_{\mathbb{Q}} \mathbb{Q}[\sqrt[3]{2}]$.

Definition 1. Let $n \in \mathbb{Z}^+$. Denote $[n] := \{1, 2, \ldots, n\}$.

Theorem 2. Let K/k be a Galois extension of finite degree n. Then $K \otimes_k K \cong K^n$, as k-algebras.

Proof. As K/k is a Galois extension of finite degree n, there exists a k-irreducible polynomial $f(x) \in k[x]$ such that $K \cong k[x]/(f(x))$. By calg3p2, we have that:

$$k[x]/(f(x)) \otimes_k K \cong K[x]/(f(x)).$$

Now as K is a splitting field for $f(x)$, we may write $f(x) = (x - a_1) \cdots (x - a_n) \in K[x]$, for distinct $a_1, \ldots, a_n \in K$. We have that:

$$\begin{aligned} K[x]/(f(x)) &\cong \prod_{i=1}^{n} K[x]/(x - a_i) \quad (1) \\ &\cong \prod_{i=1}^{n} K \cong K^n. \quad (2) \end{aligned}$$

Here, (1) follows from the Chinese Remainder Theorem and the fact that K is a splitting field for $f(x)$. The result follows. □

Corollary 3. $\mathbb{Q}[\sqrt{m}] \otimes_{\mathbb{Q}} \mathbb{Q}[\sqrt{m}] \cong \mathbb{Q}[\sqrt{m}] \times \mathbb{Q}[\sqrt{m}]$ as $\mathbb{Q}$-algebras.

Proof. We apply Theorem 2 with $K = \mathbb{Q}[\sqrt{m}]$ and $k = \mathbb{Q}$. The result follows. □

Remark 4. Let K/k be a Galois extension of finite degree n, and let $f(x) \in k[x]$ be a k-irreducible polynomial realizing $k[x]/(f(x)) \cong K$. We construct an explicit isomorphism realizing $K[x]/(f(x)) \cong k[x]/(f(x)) \otimes_k K$. Let $\varphi : K[x] \to k[x]/(f(x)) \otimes_k K$ by the k-algebra homomorphism, which is determined by $\varphi(c) = 1 \otimes_k c$ if $c \in K$ and $\varphi(x) = x \otimes_k 1$. We note that $k[x]/(f(x))$ has a basis $\{\overline{1}, \overline{x}, \ldots, \overline{x^{n-1}}\}$, and let $\{b_1, \ldots, b_n\}$ be a k-basis for K. So $k[x]/(f(x)) \otimes_k K$ is generated by:

$$\{\overline{x^i} \otimes_k b_j : i \in [n-1] \cup \{0\}, j \in [n]\}.$$

Observe that for each $i \in [n-1] \cup \{0\}$ and each $j \in [n]$:

$$\begin{aligned} \varphi(b_j x^i) &= \varphi(b_j)\varphi(x^i) \\ &= (1 \otimes_k b_j) \cdot (\overline{x^i} \otimes_k 1) \\ &= (\overline{x^i} \otimes_k b_j). \end{aligned}$$

So φ is surjective. Now observe that $\ker(\varphi)$ contains $f(x)$, and so $(f(x)) \subset \ker(\varphi)$. We now show that $\ker(\varphi) \subset (f(x))$. Let $p(x) \in \ker(\varphi)$. By the division algorithm, we may write $p(x) = f(x) \cdot q(x) + r(x)$ for some $q(x), r(x) \in K[x]$ where $\deg(r(x)) < \deg(f(x)) = n$. As φ is a k-algebra homomorphism, we have that:

$$\begin{aligned} \varphi(p(x)) &= \varphi(f(x) \cdot q(x) + r(x)) \\ &= \varphi(f(x)) \cdot \varphi(q(x)) + \varphi(r(x)) \\ &= 0 \cdot \varphi(q(x)) + \varphi(r(x)) \\ &= \varphi(r(x)). \end{aligned}$$

As $\deg(r(x)) < \deg(f(x)) = n$, we have that $\varphi(r(x)) = 0$ if and only if $r(x) = 0$. Thus, $\ker(\varphi) \subset (f(x))$. And so we conclude that $\ker(\varphi) = (f(x))$. Thus, the induced map $\overline{\varphi} : K[x]/(f(x)) \to k[x]/(f(x)) \otimes_k K$ is a well-defined isomorphism.

Remark 5. Let $R_1, \ldots, R_n$ be commutative, unital rings, and let:

$$R = \prod_{i=1}^{n} R_i.$$

We observe that for each $i \in [n]$, $e_i \in R$ (the element where all coordinates are 0, except for the ith coordinate which is 1) is an idempotent. Clearly, $e_i^2 = e_i$. Furthermore, observe that $e_i R \cong R_i$.

In the setting of Theorem 2 where we have $k[x]/f(x) \otimes_k K \cong K^n$, the standard basis vectors $e_1, \ldots, e_n \in K^n$ are the desired idempotents, whose preimages in $k[x]/f(x) \otimes_k K$ induce the direct decomposition. By the Chinese Remainder Theorem, there exist polynomials $p_1, \ldots, p_n \in K[x]$ where $p_i(a_j) = \delta_{ij}$. Denote $\overline{p_i}$ to be the projection of p_i in $K[x]/(f(x))$. The map $\overline{p_i} \mapsto e_i$ induces a k-algebra isomorphism of $K[x]/(f(x)) \cong K^n$. The preimages of the $\overline{p_i}$ in $k[x]/f(x) \otimes_k K$ are the desired idempotents.

Example 6. We first compute the idempotents in $\mathbb{Q}[\sqrt{m}] \otimes_{\mathbb{Q}} \mathbb{Q}[\sqrt{m}]$ that induce the direct decomposition. We begin working in $\mathbb{Q}[\sqrt{m}][x]/(x^2 - m)$. The elements of $\mathbb{Q}[\sqrt{m}][x]/(x^2 - m)$ are of the form $\overline{ax + b}$, with the relation that $x^2 = m$. Suppose that $\overline{ax + b} \in \mathbb{Q}[\sqrt{m}][x]/(x^2 - m)$ is an idempotent. Considering $(\overline{ax + b})^2 = \overline{ax + b}$, we obtain the following relations:

$$\begin{aligned} ma^2 + b^2 &= b \\ 2abx &= ax. \end{aligned}$$

First consider the relation that $2abx = ax$. If $a \neq 0$, then $b = 1/2$. Applying this to the first relation: $ma^2 + b^2 = b$, we obtain that:

$$a = \pm \frac{1}{2\sqrt{m}}.$$

So the idempotents in $\mathbb{Q}[\sqrt{m}][x]/(x^2 - m)$ are of the form:

$$\overline{\pm \frac{1}{2\sqrt{m}} x + \frac{1}{2}}.$$

These correspond to the following idempotents in $\mathbb{Q}[\sqrt{m}] \otimes_{\mathbb{Q}} \mathbb{Q}[\sqrt{m}]$:

$$\left(1 \otimes_{\mathbb{Q}} \frac{1}{2}\right) + \left(\sqrt{m} \otimes_{\mathbb{Q}} \frac{1}{2\sqrt{m}}\right).$$

Example 7. We compute an idempotent $e \neq 0, 1$ in $\mathbb{Q}[\sqrt[3]{2}] \otimes_{\mathbb{Q}} \mathbb{Q}[\sqrt[3]{2}]$. We note that $1, 2^{1/3}, 2^{2/3}$ are linearly independent in $\mathbb{Q}[\sqrt[3]{2}]$. So $\langle 1, 2^{1/3}, 2^{2/3} \rangle$ is a $\mathbb{Q}$-subspace of $\mathbb{Q}[\sqrt[3]{2}]$. Furthermore, we observe the following relations:

$$\begin{aligned} 1 \cdot 2^{1/3} &= 2^{1/3} \\ 1 \cdot 2^{2/3} &= 2^{2/3} \\ 2^{1/3} \cdot 2^{2/3} &= 2 \cdot 1. \end{aligned}$$

So $\langle 1, 2^{1/3}, 2^{2/3} \rangle$ is closed under products, and therefore also a sub-ring of $\mathbb{Q}[\sqrt[3]{2}]$. Thus, $\langle 1, 2^{1/3}, 2^{2/3} \rangle$ is a sub-algebra of $\mathbb{Q}[\sqrt[3]{2}]$. For that reason, we hope that the calculations simplify. So we attempt to check whether there is an idempotent of the following form:

$$e = a(1 \otimes_{\mathbb{Q}} 1) + b(2^{1/3} \otimes_{\mathbb{Q}} 2^{1/3}) + c(2^{2/3} \otimes 2^{2/3}).$$

If such an idempotent exists, then the equation $e^2 = e$ yields the relations:

$$\begin{aligned} a^2 + 8bc &= a \\ 2ab + 4c^2 &= b \\ 2ac + b^2 &= c. \end{aligned}$$

Using a computer algebra system, we find that there are four solutions to this system of equations:

$$\begin{aligned}
&a = b = c = 0 \\
&a = 1, b = c = 0 \\
&a = \frac{1}{3}, b = \frac{1}{3 \cdot 2^{2/3}}, c = \frac{1}{6 \cdot 2^{1/3}} \\
&a = \frac{2}{3}, b = -\frac{1}{3 \cdot 2^{2/3}}, c = -\frac{1}{6 \cdot 2^{1/3}}.
\end{aligned}$$

So the following elements are idempotents of $\mathbb{Q}[\sqrt[3]{2}] \otimes_{\mathbb{Q}} \mathbb{Q}[\sqrt[3]{2}]$:

$$\begin{aligned}
e_1 &= \frac{1}{3}(1 \otimes_{\mathbb{Q}} 1) + \frac{1}{3 \cdot 2^{2/3}} \cdot (2^{1/3} \otimes_{\mathbb{Q}} 2^{1/3}) + \frac{1}{6 \cdot 2^{1/3}} \cdot (2^{2/3} \otimes 2^{2/3}) \text{ and} \\
e_2 &= \frac{2}{3}(1 \otimes_{\mathbb{Q}} 1) - \frac{1}{3 \cdot 2^{2/3}} \cdot (2^{1/3} \otimes_{\mathbb{Q}} 2^{1/3}) - \frac{1}{6 \cdot 2^{1/3}} \cdot (2^{2/3} \otimes 2^{2/3}).
\end{aligned}$$