Commutative Algebra
Assignment 3
Problem 2

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2. Let $\mathbb{F}$ be a field. Suppose that A and B are $\mathbb{F}$-algebras and that $B = \mathbb{F}[b]$ is generated as an $\mathbb{F}$-algebra by a single element $b \in B$.

(a) Show that $A \otimes_{\mathbb{F}} B \cong A[x]/\min_{b,\mathbb{F}}(x)$.

(b) Restrict now to the case where A and B are fields. Give an example where $A \otimes_{\mathbb{F}} B$ has nonzero nilpotent elements, and another example where $A \otimes_{\mathbb{F}} B$ $(\neq 0)$ has no nonzero nilpotent ideals.

(a) In this proof, we show there is an invertible $\mathbb{F}$-algebra homomorphism from $A \otimes_{\mathbb{F}} B$ to $A[x]/\min_{b,\mathbb{F}}(x)$. To do so, we make use of the universal property of quotients and the universal property of tensor products.

Let $\pi : A[x] \to A[x]/\min_{b,\mathbb{F}}(x)$ be the canonical projection and let $\phi : A[x] \to A \otimes_{\mathbb{F}} B$ be given by

$$\phi\left(\sum_{k=0}^{n} a_k x^k\right) = \sum_{k=0}^{n} a_k \otimes b^k$$

In order to appeal to the universal property of quotients, we perform a quick computation. Let $f_0, ..., f_k \in \mathbb{F}$ such that $\min_{b,\mathbb{F}}(x) = \sum_{k=0}^{n} f_k x^k$. Then:

$$\begin{aligned}
\phi(\min_{b,\mathbb{F}}(x)) &= \phi\left(\sum_{k=0}^{n} f_k x^k\right) \\
&= \sum_{k=0}^{n} f_k \otimes b^k && \text{(Def. of } \phi) \\
&= \sum_{k=0}^{n} 1 \otimes f_k b^k && (f_k \in \mathbb{F}) \\
&= 1 \otimes \sum_{k=0}^{n} f_k b^k && \text{(Bilinearity of } \otimes) \\
&= 1 \otimes \min_{b,\mathbb{F}}(b) \\
&= 1 \otimes 0 \\
&= 0
\end{aligned}$$

Thus, $(\min_{b,\mathbb{F}}(x)) \subseteq \ker(\phi)$, so by the universal property of quotients, there is an $\mathbb{F}$-module homomorphism $\overline{\phi} : A[x]/\min_{b,\mathbb{F}}(x) \to A \otimes_{\mathbb{F}} B$ such that $\overline{\phi} \circ \pi = \phi$.

Now consider the map $s : A \times B \to A[x]$ determined by $s(a, b^k) = ax^k$ (not every element of B is of the form b^k, but we extend this to an $\mathbb{F}$-algebra homomorphism). Define $\varphi = \pi \circ s$.

In order to appeal to the universal property of tensor products, observe that φ is bilinear since s is bilinear (by def. of $A[x]$) and π is linear. Let $\iota : A \times B \to A \otimes_{\mathbb{F}} B$ be the standard insertion of generators. By the universal property of tensor products, there is an $\mathbb{F}$-algebra homomorphism $\overline{\varphi} : A \otimes_{\mathbb{F}} B \to A[x]/\min_{b,\mathbb{F}}(x)$ such that $\overline{\varphi} \circ \iota = \varphi$.

All necessary maps are now defined so we will now show that $\overline{\varphi}$ and $\overline{\phi}$ are inverses. For this purpose, it suffices to show that $\overline{\varphi}$ is surjective and that $\overline{\phi} \circ \overline{\varphi} = id_{A\otimes_{\mathbb{F}} B}$. This suffices for two reasons. First, every left invertible function is injective, so a left invertible surjection is a bijection. Second, a left inverse of a bijective function is a full inverse.

To see that $\overline{\varphi}$ is surjective, consider $\overline{\varphi}|_{\mathrm{im}(\iota)}$. According to our definitions, we have

$$\mathrm{im}(\overline{\varphi}|_{\mathrm{im}(\iota)}) = \mathrm{im}(\overline{\varphi} \circ \iota) = \mathrm{im}(\varphi) = \mathrm{im}(\pi \circ s)$$

Since π is surjective, we need only show that s is surjective. However, this is immediate since for any $\sum_{k=0}^{n} a_k x^k \in A[x]$, the definition of s yields

$$s\left(\sum_{k=0}^{n} a_k b^k\right) = \sum_{k=0}^{n} s\left(a_k b^k\right) = \sum_{k=0}^{n} a_k x^k$$

Hence, $\overline{\varphi}$ is surjective.

To see that $\overline{\phi} \circ \overline{\varphi} = id_{A\otimes_{\mathbb{F}} B}$, first consider $\overline{\phi} \circ \overline{\varphi} \circ \iota$. By our definitions, we have

$$\overline{\phi} \circ \overline{\varphi} \circ \iota = \overline{\phi} \circ \varphi = \overline{\phi} \circ \pi \circ s = \phi \circ s$$

We will now show that $\phi \circ s = \iota$. Since $\phi \circ s$ is an $\mathbb{F}$-algebra homomorphism, it suffices to show that for all $a \in A$ and $k \in \mathbb{N}$, $(\phi \circ s)(a, b^k) = a \otimes b^k$. Again, this follows directly from the definition of ϕ and s: $(\phi \circ \iota)(a, b^k) = \phi(ax^k) = a \otimes b^k = \iota(a, b^k)$. Therefore, $\overline{\phi} \circ \overline{\varphi} \circ \iota = \iota$. We then have that for any simple tensor $a \otimes b^k \in A \otimes_{\mathbb{F}} B$, $(\overline{\phi} \circ \overline{\varphi})(a \otimes b^k) = (\overline{\phi} \circ \overline{\varphi} \circ \iota)(a, b^k) = \iota(a, b^k) = a \otimes b^k$. Such tensors form a basis for $A \otimes_{\mathbb{F}} B$, so $\overline{\phi} \circ \overline{\varphi} = id_{A\otimes_{\mathbb{F}} B}$.

By our previous discussion, we conclude that $\overline{\varphi}$ is an invertible $\mathbb{F}$-algebra homomorphism, so $A \otimes_{\mathbb{F}} B \cong A[x]/\min_{b,\mathbb{F}}(x)$.

(b) Let $\mathbb{F}$ be any transcendental simple extension of $\mathbb{F}_2$, say $\mathbb{F} = \mathbb{F}_2(t)$. The polynomial $x^2 - t$ is irreducible over $\mathbb{F}$ (since $\sqrt{t} \notin \mathbb{F}$) and is monic, so $\min_{\sqrt{t},\mathbb{F}}(x) = x^2 - t$. By part (a), we therefore have

$$\mathbb{F}(\sqrt{t}) \otimes_{\mathbb{F}} \mathbb{F}(\sqrt{t}) \cong \mathbb{F}(\sqrt{t})[x]/(x^2 - t)$$

But $x^2 - t$ is reducible over $\mathbb{F}(\sqrt{t})$! In particular, since $\mathbb{F}$ has characteristic 2, $(x - \sqrt{t})^2 = x^2 - 2x\sqrt{t} + t = x^2 - t$, so we actually have

$$\mathbb{F}(\sqrt{t}) \otimes_{\mathbb{F}} \mathbb{F}(\sqrt{t}) \cong \mathbb{F}(t)[x]/(x - \sqrt{t})^2$$

$\mathbb{F}(t)[x]/(x - \sqrt{t})^2$ has nonzero element $x - \sqrt{t}$ which squares to zero, so we are done with the first example.

Consider $\mathbb{Q}$ as a $\mathbb{Q}$-algebra under its usual ring structure and $\mathbb{Q}$-action given by multiplication. Then $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ (is isomorphic to $\mathbb{Q}$ since generally $R \otimes_R S \cong S$, but let's use part **(a)**) is isomorphic to $\mathbb{Q}[x]/\min_{1,\mathbb{Q}}(x)$ since $\mathbb{Q}$ is generated as a $\mathbb{Q}$-algebra by 1. Since $1 \in \mathbb{Q}$, this extension is trivial and we arrive at $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q} \cong \mathbb{Q}$. Since $\mathbb{Q}$ has no nonzero nilpotent elements and is itself nonzero, we are done.