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Commutative Algebra
Assignment 3

Problem 1. Lem M be a finitely generated R-module. Show that $R/I \otimes_R M = 0$ if and only if there exists $i \in I$ such that $(1+i)M = 0$ in two different ways:

(1) using Nakayama's lemma.

(2) avoiding Nakayama's lemma.

Lemma 1. *Let M be a R-module, then $R/I \otimes_R M$ is isomorphic to M/IM.*

Proof. Consider the short exact sequence

$$0 \longrightarrow I \xrightarrow{\iota} R \xrightarrow{p} R/I \longrightarrow 0$$

where ι is the inclusion map and p is the natural projection $r \mapsto r + I$. Recall that $(-) \otimes_R M$ is a right-exact functor from R-modules to R-modules. It follows that

$$I \otimes_R M \xrightarrow{\iota \otimes \mathrm{id}} R \otimes_R M \xrightarrow{p \otimes \mathrm{id}} R/I \otimes_R M \longrightarrow 0$$

is exact. Now, $R \otimes_R M$ is isomorphic to M. In particular, the map $\Phi : R \otimes_R M \to M$ that is defined on simple tensors as the map $r \otimes m \mapsto rm$ gives such an isomorphism. So, denote $\varphi = \Phi \circ (\iota \otimes \mathrm{id})$ and $\psi = (p \otimes \mathrm{id}) \circ \Phi$ then

$$I \otimes_R M \xrightarrow{\varphi} M \xrightarrow{\psi} R/I \otimes_R M \longrightarrow 0$$

is exact. Observe that for $i \in I$ and $m \in M$, $\varphi(i \otimes m) = im$ so the image of φ is IM. That is, $\ker \psi = IM$. Furthermore, exactness at $R/I \otimes_R M$ means ψ is surjective. It follows from the fundamental isomorphism theorems that $M/IM \cong R/I \otimes_R M$. $\square$

Proposition 2. *Suppose M is a finitely generated R-module and $I \subseteq R$ is an ideal. If $R/I \otimes_R M = 0$ then there exists $i \in I$ such that $(1+i)M = 0$.*

Proof (using Nakayama's lemma directly). Suppose $R/I \otimes_R M = 0$, then by Lemma 1 we also have $M/IM = 0$. However, this means $M = IM$. Since M is finitely generated, we may apply Nakayama's lemma, which assets that there exists $i \in I$ such that $(1+i)M = 0$. $\square$

Proof (avoiding directly citing Nakayama's lemma). Recall that for two finitely generated R-modules M and N, $N \otimes_R M = 0$ if and only if $\mathrm{Ann}(N) + \mathrm{Ann}(M) = R$. By assumption, M is finitely generated and R/I is also finitely generated since $1 + I$ is a generator. Then $R/I \otimes_R M = 0$ implies that $\mathrm{Ann}(R/I) + \mathrm{Ann}(M) = R$.

Now if $r \in R$ annihilates R/I, then $r(1 + I) = I$. However, that means that $r \in I$. Clearly, every element of I annihilates R/I, so it follows that $\mathrm{Ann}(R/I) = I$. Then, the fact that $\mathrm{Ann}(R/I) + \mathrm{Ann}(M) = R$ means there is some element $i \in I$ and $x \in \mathrm{Ann}(M)$ such that $i + x = 1$. It follows that $x = 1 + (-i)$. Since $-i \in I$, then we are done. $\square$

Corollary 3. *Let M be a finitely generated R module and $I \subseteq R$ be an ideal. Then $R/I \otimes_R M = 0$ if and only if there exists $i \in I$ such that $(1+i)M = 0$.*

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Proof. The forward direction is given by Proposition 3, so suppose there exists $i \in I$ such that $(1+i)M = 0$. To prove that $R/I \otimes_R M = 0$, it is sufficient to prove that the simple tensors are 0. So let $r + I \in R/I$ and $m \in M$, then

$$\begin{aligned}(r+I) \otimes m &= ((1+i)(r+I)) \otimes m \\ &= (r+I) \otimes (1+i).m \\ &= (r+I) \otimes 0 \\ &= 0.\end{aligned}$$

□