Commutative Algebra- HW2

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Problem 9. This problem involves Nakayama's Lemma. Suppose that $(R, \mathfrak{m})$ is a local ring with maximal ideal $\mathfrak{m}$, and that M is a finitely generated R-module.

(a) Show that a subset $F \subseteq M$ is a generating set if and only if $F/\mathfrak{m}$ is a generating set for the $R/\mathfrak{m}$-vector space $M/\mathfrak{m}M$. Conclude that all minimal generating sets for M have the same size.

(b) Show that a homomorphism $\varphi : M \to N$ between finitely generated R-modules is surjective if and only if the induced map $\varphi_{\mathfrak{m}} : M/\mathfrak{m}M \to N/\mathfrak{m}N$ is surjective.

Theorem 1. Suppose that $(R, \mathfrak{m})$ is a local ring with maximal ideal $\mathfrak{m}$, and that M is a finitely generated R-module. Let $F \subset M$. We have that F is a generating set of M if and only if $F/\mathfrak{m}$ is a generating set for the $R/\mathfrak{m}$-vector space $M/\mathfrak{m}M$.

We break the proof of this theorem into a series of propositions.

Proposition 2. Suppose that $(R, \mathfrak{m})$ is a local ring with maximal ideal $\mathfrak{m}$, and that M is a finitely generated R-module. Let $F \subset M$. If that F is a generating set of M, then $F/\mathfrak{m}$ is a generating set for the $R/\mathfrak{m}$-vector space $M/\mathfrak{m}M$.

Proof. Denote $F = \{f_1, \ldots, f_n\}$. Suppose first that F generates M. Consider the natural projection map $\pi : M \to M/\mathfrak{m}M$. Observe that $\pi(F) \subset M/\mathfrak{m}M$. So $\langle \pi(F) \rangle \subset M/\mathfrak{m}M$. We now show that $\langle \pi(F) \rangle \supset M/\mathfrak{m}M$. Let $x + \mathfrak{m}M \in M/\mathfrak{m}M$, and denote $x \in M$ to be a preimage of $x + \mathfrak{m}M$ under the projection map. As F generates M, we may write x as an R-linear combination in terms of F:

$$x = \sum_{i=1}^{n} r_i f_i,$$

where $r_1, \ldots, r_n \in R$. It follows that:

$$\begin{aligned} x + \mathfrak{m}M &= \pi(x) \\ &= \pi\left(\sum_{i=1}^{n} r_i f_i \right) \\ &= \sum_{i=1}^{n} (r_i + \mathfrak{m}) \pi(f_i). \end{aligned}$$

So $x + \mathfrak{m}M \in \langle \pi(F) \rangle$. As $x + \mathfrak{m}M$ was arbitrary, it follows that $\langle \pi(F) \rangle \supset M/\mathfrak{m}M$. So $\pi(F) = F/\mathfrak{m}$ generates $M/\mathfrak{m}M$, as desired. □

We recall Corollary 2.7 from Atiyah-Macdonald.

Lemma 3 (Corollary 2.7, Atiyah-Macdonald)**.** Let M be a finitely generated R-module, let $N \leq M$ be a sub-module, and let I be an ideal of the Jacobson radical $J(R)$. If $M = IM + N$, then $M = N$.

Proof. We note that:

$$I(M/N) = (IM + N)/N = M/N.$$

The last equality follows from the assumption that $IM + N = M$. As $I(M/N) = M/N$, we have by Nakayama's Lemma that $M/N = 0$. So $M = N$. □

Proposition 4. Suppose that $(R, \mathfrak{m})$ is a local ring with maximal ideal $\mathfrak{m}$, and that M is a finitely generated R-module. Let $F \subset M$. If $F/\mathfrak{m}$ is a generating set for the $R/\mathfrak{m}$-vector space $M/\mathfrak{m}M$, then F is a generating set of M.

Proof. Denote $F = \{f_1, \ldots, f_n\}$, and let $N = \langle F \rangle$ be the submodule of M generated by F. As $F/\mathfrak{m}$ generates $M/\mathfrak{m}M$, it follows that:

$$M/\mathfrak{m}M = \sum_{i=1}^{n} (R/\mathfrak{m})(f_i + \mathfrak{m}M).$$

Thus,

$$\begin{aligned} M/\mathfrak{m}M &= \left\{ \left(\sum_{i=1}^{n} r_i f_i \right) + \mathfrak{m}M : r_1, \ldots, r_n \in R \right\} \\ &= (N + \mathfrak{m}M)/\mathfrak{m}M. \end{aligned}$$

So $M = N + \mathfrak{m}M$. By Corollary 2.7 from Atiyah-Macdonald, we have that $M = N$. So F is a generating set for M, as desired. □

We obtain the following corollary to Theorem 1.

Corollary 5. Suppose that $(R, \mathfrak{m})$ is a local ring with maximal ideal $\mathfrak{m}$, and that M is a finitely generated R-module. All minimal generating sets have size $\dim_{R/\mathfrak{m}}(M/\mathfrak{m}M)$.

Proof. Let F be a minimal generating set for M. We have by Theorem 1 that $F/\mathfrak{m}$ is a generating set for $M/\mathfrak{m}M$. We note that if $F/\mathfrak{m}$ is a basis for $M/\mathfrak{m}M$, then we are done. So we claim that $F/\mathfrak{m}$ is a minimal generating set for $M/\mathfrak{m}M$. Suppose to the contrary that $F'/\mathfrak{m} \subsetneq F/\mathfrak{m}$ is a smaller generating set for $M/\mathfrak{m}M$. Then by Theorem 1, $F' \subsetneq F$ is a generating set for M, contradicting the minimality of F. So $F/\mathfrak{m}$ is indeed a basis for $M/\mathfrak{m}M$, as desired. □

Theorem 6. Suppose that $(R, \mathfrak{m})$ is a local ring with maximal ideal $\mathfrak{m}$. Let M, N be finitely generated R-modules, and let $\varphi : M \to N$ be an R-module homomorphism. We have that φ is surjective if and only if the induced map $\varphi_{\mathfrak{m}} : M/\mathfrak{m}M \to N/\mathfrak{m}N$ is surjective.

Proof. Let $\pi_M : M \to M/\mathfrak{m}M, \pi_N : N \to N/\mathfrak{m}N$ be the natural projection maps. We note that $\pi_N \circ \varphi : M \to N/\mathfrak{m}N$ is an R-module homomorphism. We claim that $\mathfrak{m}M \subset \ker(\pi_N \circ \varphi)$. As φ is an R-module homomorphism, we have that:

$$(\pi_N \circ \varphi)(\mathfrak{m}M) = \pi_N(\mathfrak{m}\varphi(M)).$$

Now we note that $\mathfrak{m}\varphi(M) \subset \mathfrak{m}N$. So $\pi_N(\mathfrak{m}\varphi(M)) = 0$. Thus, $\mathfrak{m}M \subset \ker(\pi_N \circ \varphi)$. So by the universal property of quotients, there exists an induced map $\varphi_{\mathfrak{m}} : M/\mathfrak{m}M \to N/\mathfrak{m}N$ which satisfies:

$$\pi_N \circ \varphi = \varphi_{\mathfrak{m}} \circ \pi_M.$$

Let F be a generating set for M. As F is a generating set for M, φ is determined by $\varphi(F)$. By Theorem 1, $F/\mathfrak{m}$ is a generating set for $M/\mathfrak{m}M$. Suppose first that φ is surjective. So $\langle \varphi(F) \rangle = N$. By Theorem 1, $\varphi(F)/\mathfrak{m}$ is a generating set for $N/\mathfrak{m}N$. As $\varphi_{\mathfrak{m}}(F/\mathfrak{m}) = \varphi(F)/\mathfrak{m}$, it follows that $\varphi_{\mathfrak{m}}$ is surjective.

Conversely, suppose that $\varphi_{\mathfrak{m}}$ is surjective. As $F/\mathfrak{m}$ is a generating set for $M/\mathfrak{m}M$ and $\varphi_{\mathfrak{m}}$ is surjective, we have that $\varphi_{\mathfrak{m}}(F/\mathfrak{m}) = \varphi(F)/\mathfrak{m}$ is a generating set for $N/\mathfrak{m}N$. By Theorem 1, $\varphi(F)$ is a generating set for N. Thus, φ is surjective. □