COMMUTATIVE ALGEBRA HOMEWORK 2

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Problem (8)**.** Let M be an R-module over a commutative ring R.

(a) Show that $J(M)$ consists of the *nongenerators* of M: i.e. $m \in J(M)$ if and only if $M = \langle S \cup \{m\} \rangle$ implies $M = \langle S \rangle$.

(b) Exhibit an example to show that infinitely many elements from $J(M)$ might not be cancellable from a generating set.

(c) Show that if M is finitely generated and $P \subseteq J(M)$, then $M = N + P$ implies $M = N$. (This means any set of elements of $J(M)$ may be cancelled from a generating set of a finitely generated module.) In particular, show that if $I \subseteq J(R)$, M is finitely generated, and $M = N + IM$, then $M = N$.

Claim (a)**.** For an R-module M, $J(M)$ is precisely the set of nongenerators of M.

Proof. Recall that $J(M)$ is the intersection of all maximal submodules of M.

Let $m \in M$ and suppose $m \notin J(M)$. For this to be the case, we must have $J(M) \lneq M$ and thus M contains at least one maximal submodule $N \prec M$, necessarily with $m \notin N$. But this means that $N \lneq \langle N \cup \{m\} \rangle = M$, thus m is essential in the generating set $N \cup \{m\}$, i.e. m is *not* a nongenerator.

Conversely, if $m \in M$ is *not* a nongenerator, then there is some set $S \subseteq M$ so that $S \cup \{m\}$ generates M (i.e., $\langle S \rangle + \langle m \rangle = M$), but $\langle S \rangle \lneq M$. Note that this implies $m \notin \langle S \rangle$, and that $K = \langle S \rangle \cap \langle m \rangle$ is strictly below $\langle S \rangle$ and $\langle m \rangle$. Consider then the isomorphic intervals $[K, \langle m \rangle]$ and $[\langle S \rangle, M]$ in the submodule lattice of M. Since $\langle m \rangle$ is, in particular, a finitely generated (sub-)module, K is contained in a maximal submodule below $\langle m \rangle$: $K \leq N \prec \langle m \rangle$. Then by perspective isomorphism, we obtain a maximal submodule N' with $\langle S \rangle \leq N' \prec M$, and $m \notin N'$. Thus, we have exhibited a maximal submodule not containing m, and $m \notin J(M)$. $\square$

Example (b). We demonstrate a module M where the omittance of infinitely many elements of $J(M)$ from a generating set no longer generates M. Simply consider $\mathbb{Q}$ as a $\mathbb{Z}$-module; Note that $\mathbb{Q}$ is not finitely generated, and moreover $\mathbb{Q}$ has no maximal submodules, so $J(\mathbb{Q}) = \mathbb{Q}$. Thus any generating set for $\mathbb{Q}$ is an infinite set of elements of $J(\mathbb{Q})$; removing all of them leaves $\varnothing$, which evidently does not generate $\mathbb{Q}$.

Claim (c)**.** If M is finitely generated and $P \leq J(M)$ is a fixed submodule, then $N + P = M$ implies $N = M$ for any submodule N.

Proof. Toward the contrapositive, suppose N was an arbitrary *proper* submodule. Then since M is finitely generated, N is contained in a maximal submodule $N \leq N' \prec M$. On the other hand, $P \leq J(M)$ which is contained in all maximal submodules of M, so $P \leq N'$. Then, $N + P \leq N'$ as well. $\square$

Corollary (c)**.** If M is finitely generated, any set of members of $J(M)$ may be freely removed from a generating set for M.

Proof. If S and P are sets with $P \subseteq J(M)$ such that $M = \langle S \cup P \rangle$, then $\langle P \rangle \leq J(M)$ and $\langle S \rangle + \langle P \rangle = M$, so it must be the case that $\langle S \rangle = M$. $\square$

Corollary (c)**.** If M is finitely generated, and $I \leq J(R)$ is a fixed ideal, then $M = N + IM$ implies $N = M$ for any submodule N.

Proof. This follows from the fact that $IM \leq J(R)M \leq J(M)$; to see the second inclusion, take any maximal submodule N and note that M/N is simple; since $J(R)$ annihilates all simple R-modules, we have $J(R)(M/N) = 0$ and thus $J(R)M \leq N$. $\square$