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6. Here we consider Spec(R) as a topological space (the primes equipped with the Zariski topology) and as an ordered set (the primes equipped with the inclusion order).

(a) Show that the inclusion order on the prime ideals can be recovered from the topology of Spec(R).

(b) Show that conversely, if R is a Noetherian ring, then the topology of Spec(R) can be determined from the inclusion order on the prime ideals.

(c) Show that if R is not Noetherian, then the topology of Spec(R) may not be recoverable from the inclusion order on the primes.

Proof.

(a) Let $\mathbf{p}$ and $\mathbf{q}$ be prime ideals of the commutative ring R. Note that by the definition of Spec(R), the closed sets are the vanishing sets $V(S)$ for some $S \subseteq R$. That is, they are given by sets of primes which contain a given subset S. So we have that

$$\overline{\{\mathbf{q}\}} = \bigcap_{S \subseteq \mathbf{q}} V(S).$$

Note that if $\mathbf{p} \in \overline{\{\mathbf{q}\}}$, then $\mathbf{p} \in V(S)$ for all $S \subseteq \mathbf{q}$, hence in particular $\mathbf{p} \in V(\mathbf{q})$. Also, if $\mathbf{p} \in V(\mathbf{q})$, then $\mathbf{q} \subseteq \mathbf{p}$ so that $S \subseteq \mathbf{p}$ for all $S \subseteq \mathbf{q}$. Thus, if $\mathbf{p} \in V(\mathbf{q})$ then $\mathbf{p} \in \overline{\{\mathbf{q}\}}$ as well, and we have that $\overline{\{\mathbf{q}\}} = V(\mathbf{q})$.

In summary, we have that the following are equivalent for $\mathbf{p}, \mathbf{q} \in \text{Spec}(R)$:

- $\mathbf{p} \in \overline{\{\mathbf{q}\}}$
- $\mathbf{p} \in V(\mathbf{q})$
- $\mathbf{q} \subseteq \mathbf{p}$.

Thus, given the topology of Spec(R) we can decide the inclusion relation for any two prime ideals.

(b) Let $V = V(S)$ for some $S \subseteq R$ be an arbitrary closed set of Spec(R). We always have that

$$V(S) = V(\langle S \rangle) = V(\sqrt{\langle S \rangle}) = V(\bigcap_{\substack{S \subseteq \mathbf{p} \\ \mathbf{p} \text{ prime}}} \mathbf{p}).$$

Since R is Noetherian, by Homework 2 Problem 5 this intersection can be taken over the finitely many prime ideals $\mathbf{p}_1, \ldots, \mathbf{p}_n$ which are minimal with respect to containing S. We have then that

$$V(S) = V(\cap_{i=0}^{n} \mathbf{p}_i)$$

but since this intersection is finite we may equivalently take the union over the $V(\mathbf{p}_i)$

$$V(\cap_{i=0}^{n} \mathbf{p}_i) = \cup_{i=1}^{n} V(\mathbf{p}_i)$$

so that every closed set $V(S)$ is the union of finitely many closed sets of the form $V(\mathbf{p})$ for some prime ideal $\mathbf{p}$.

Each $V(\mathbf{p})$ is the set of all primes which contain $\mathbf{p}$. But this is exactly the *principal filter* [1] generated by $\mathbf{p}$ in the inclusion order of prime ideals!

Hence, if R is Noetherian then we have that the following are equivalent for any subset $V \subseteq \mathrm{Spec}(R)$:

- V is closed in the Zariski topology.
- V is a finite union of principle filters in the inclusion order.

(c) Consider the field $\mathbb{F}_2$ and the ring $R = \mathbb{F}_2^\omega$ of countably many copies of ω. Then R is not Noetherian, as for instance the chain of ideals

$$\langle (0,0,0,\ldots)\rangle \subseteq \langle (1,0,0,\ldots)\rangle \subseteq \langle (1,1,0,\ldots)\rangle \ldots$$

is an ascending chain of ideals that never stabilizes.

Note that we can think an element $(b_n)_{n\in\omega} \in R$ as a characteristic function $\chi_B : \omega \to \mathbb{F}_2$ for some subset $B \subseteq \omega$, regarding $b_n = 1$ as indicating that $n \in B$ and $b_n = 0$ as $n \notin B$. Then the zero element of R, the sequence where $b_n = 0$ for all n, corresponds to $\chi_\varnothing$, the characteristic function of the empty set. The unit element, where $b_n = 1$ for all n, is then χ_ω.

Since $a_n b_n = 1$ if and only if $a_n = b_n = 1$, the corresponding multiplication operation on characteristic functions χ_A and χ_B respectively, should produce a χ_C such that $n \in C$ if and only if $n \in A$ and $n \in B$. Hence, we must have $C = A \cap B$, so we define multiplication by $\chi_A \chi_B = \chi_{A\cap B}$.

Similarly, since $a_n + b_n = 0$ if and only if $a_n = b_n$, the addition operation should choose a set $C \subseteq \omega$ such that $n \in C$ if and only if $n \in A$ and $n \notin B$ or $n \in B$ and $n \notin A$. Hence, $\chi_A + \chi_B = \chi_{A\delta B}$ where $A\delta B = (A \cup B) \setminus (A \cap B)$ is the symmetric difference of A and B, i.e. all those things that are in exactly one of A or B.

Hence, we can think of R as the ring of characteristic functions on subsets of ω, with the above operations and distinguished elements. We will show that the topology of $\mathrm{Spec}(R)$ is not definable from the order relation on $\mathrm{Spec}(R)$.

First we claim that a subset I of R is an ideal if and only if the characteristic functions in I correspond to a collection of subsets $\mathcal{F}$ of ω whose complement $\mathcal{F}^c$ forms a filter[2]. That is, we must show that these subsets are *downwards closed*, i.e. if $\chi_B \in I$ and $A \subseteq B$ then $\chi_B \in I$, and that they are *upwards directed*, i.e. if $\chi_A \in I$ and $\chi_B \in I$ then we have $\chi_{A\cup B} \in I$.

Suppose that I is an ideal. Note then that if $\chi_B \in I$ and $A \subseteq B$, then $\chi_A\chi_B = \chi_{A\cap B} = \chi_A \in I$ since I is closed under multiplication by elements of R, so that I is

[1] A *filter* of a poset P is a nonempty subset F that is *upwards closed*, i.e if $x \in F$ and $x \leq y$ then $y \in F$, an *downward directed*, i.e. for every $x, y \in F$ there is some $z \in F$ such that $z \leq x$ and $z \leq y$. A filter is *principal* if is the set of all elements above some given $x \in F$.

[2] This is also called being an *ideal* in the inclusion poset of subsets of ω, we but we avoid the clash in terminology here.

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downwards closed. Then, if χ_A and χ_B are both in I, consider $A \cup B$. We have that $A \cup B = (A\delta B) \cup (A \cap B)$ where the right hand union is now disjoint, so $\chi_{A\cup B} = (\chi_A + \chi_B) + \chi_A\chi_B$. The right hand side is a sum of elements of I, so $\chi_{A\cup B} \in I$ and I is upwards directed.

Now, suppose that the set $\mathcal{F}$ of subsets of ω corresponding to I is downwards closed and upwards directed. Let $\chi_A, \chi_B \in I$. Then since $\mathcal{F}$ is upwards directed, $A \cup B \in \mathcal{F}$ so that $\chi_{A\cup B} \in I$. Since it is downwards closed, the symmetric difference $A\delta B$, being a subset of $A \cup B$, is also in $\mathcal{F}$, so $\chi_A + \chi_B = \chi_{A\delta B} \in I$ and I is closed under addition.

Now suppose that $\chi_B \in I$ and let $\chi_A \in R$ be arbitrary. Then since $\mathcal{F}$ is downwards closed, we have $A \cap B \in \mathcal{F}$ as it is a subset of both. Hence $\chi_A\chi_B = \chi_{A\cap B} \in I$ and I is an ideal as desired. Thus, we have that I is an ideal if and only if $\mathcal{F}$ is downwards closed and upwards directed.

We claim also that an ideal I is prime if and only if filter $\mathcal{F}^c$ is an ultrafilter[3] Hence, we wish to show that I is prime if and only if for any subset $B \subseteq \omega$, either B or $\omega \setminus B$ is in $\mathcal{F}$. First, suppose that I is prime. Then since $\chi_B\chi_{\omega\setminus B} = \chi_\varnothing \in I$, we must have at least one of χ_B or $\chi_{\omega\setminus B}$ in I, hence B or $\omega \setminus B$ is in $\mathcal{F}$ as desired. Now suppose the converse, and let $\chi_A\chi_B \in I$ for some $\chi_A\chi_B \in R$. Suppose further that $\chi_A \notin I$. Then we must have that $\chi_{\omega\setminus A} \in I$. But then $\chi_{\omega\setminus A}\chi_B \in I$ and so $\chi_A\chi_B + \chi_{\omega\setminus A}\chi_B = \chi_B \in I$. Hence I is prime.

Two different prime ideals $\mathbf{p}, \mathbf{q} \in \operatorname{Spec}(R)$ must thus be incomparable. If they were not, and say $\mathbf{p} \subseteq \mathbf{q}$, then the corresponding filter for $\mathbf{p}$ is a subset of the filter for $\mathbf{q}$, hence there would be subsets A of ω in the filter for $\mathbf{q}$ for which neither A or its complement are decided on by the filter for $\mathbf{p}$, contradicting that it is an ultrafilter. Note also that there must be infinitely many such prime ideals, as for instance there is a principal ultrafilter on ω for each element $n \in \omega$ Thus $\operatorname{Spec}(R)$ is here an infinite antichain.

Note then that the order relation on $\operatorname{Spec}(R)$ is exactly the equality relation. Any topology on $\operatorname{Spec}(R)$ defined from the order relation should be invariant under maps which preserves the order relation. In this case, any such topology must be invariant under permutations, as these are exactly the map which preserve the order relation. We show that there is no spectral topology on a space $X = \operatorname{Spec}(R)$ which is compatible with every permutation of X. To do this, we will show that these properties imply every singleton is open, hence X cannot be compact, contradicting that X is spectral.

First we will show that singletons are closed so that X is reducible. Suppose that X has a topology that is compatible with every permutation of X. Since every nontrivial ring has a maximal ideal, and for a maximal (hence prime) ideal $\mathfrak{m}$, $\{\mathfrak{m}\} = V(\mathfrak{m})$, we have that a nonempty spectral space X always has at least one closed point. The topology is compatible with permutations, hence if we apply a permutation to the closed set $\{\mathfrak{m}\}$ the resulting set is also closed. But we can map any points $x_1, x_2 \in X$ to each other via a transposition (x_1x_2), leaving all other other points invariant, hence every point in X must be closed. Since every singleton is closed, the irreducible components of X are exactly the singletons. Hence, X is reducible.

[3] A filter (of subsets of some set X) is an *ultrafilter* if for any subset $S \subseteq X$, either S or $X \setminus S$ is in the filter.

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Note that since X is reducible, if $X = Y \cup Z$ for two nonempty proper closed subsets $Y, Z \subset X$, then by taking complements relative to X, $X^c = (Y \cup Z)^c = Y^c \cap Z^c$. Note that the complements Y^c and Z^c are nonempty as both Y and Z are proper. But, $X^c = \varnothing$, so Y^c and Z^c are nonempty disjoint open sets.

Now we will produce an open singleton by manipulating Y^c and Z^c with permutations. Let $y \in Y^c$ and $z \in Z^c$. Then $y \neq z$, so consider the transposition $\sigma = (yz)$ applied to Y^c. Since the topology is invariant under permutations, σY^c is also open. But, σY^c clearly contains z, so $(\sigma Y^c) \cap Z^c = \{z\}$ is an intersection of two open sets, hence open.

Since some singleton is open, by invariance under permutations we have every singleton is open similar to the above with closed singletons. But, if every singleton is open, then X has an open cover of singletons for which no subcover can cover X, so X is not compact. Hence, no space X can be both spectral and invariant under permutations. ■