calghw2p5

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Suppose that $I \triangleleft R$ has infinitely many primes that are minimal above it.

(a) Show that I is not prime.

(b) Use (a) to show that there is an ideal properly containing I that also has infinitely many minimal primes above it.

(c) Conclude that R is not Noetherian. (Expressed more positively, any Noetherian ring has the property that every ideal I has only finitely many minimal primes containing it, hence $\sqrt{I}$ is the intersection of finitely many primes.

Proof.

(a) Since I is contained in more than prime minimal over I, at least one inclusion must be proper. Because I is a proper subset of some prime minimal over I, it cannot itself be prime.

(b) Because I is not prime, there must be two ideals J, K such that $J, K \not\subseteq I$ and $JK \subseteq I$. We may assume that $I \subsetneq J, K$ by the following argument.

Let $\widetilde{J} = J + I$. Let $\widetilde{K} = K + I$. Then we have $J \subsetneq \widetilde{J}$ and $K \subsetneq \widetilde{K}$. Also,

$$\widetilde{J}\widetilde{K} = (J + I)(K + I) \subseteq JK + I = I.$$

Then $\widetilde{J}$ and $\widetilde{K}$ are ideals with the desired properties.

For each prime p containing I we have

$$JK \subseteq I \subseteq p,$$

showing that either $J \subseteq p$ or $K \subseteq p$. Since there are infinitely many such minimal primes, either J or K must be contained in infinitely many primes minimal over I. Suppose without loss of generality that J is contained in infinitely many primes minimal over I. Let p be a prime minimal over I containing J. Any prime q such that $J \subseteq q \subsetneq p$ would also be a prime such that $I \subseteq q \subsetneq p$, contradicting the minimality of p over I. Therefore the infinitely many primes (minimal over I) containing J are also minimal over J.

(c) Let $I_0 = I$. Define a sequence $(I_0, I_1, ...)$ recursively such that I_{n+1} is some ideal properly containing I_n that is contained in infinitely many minimal primes, whose existence is guaranteed by part (b). Since all inclusions are proper, we have

$$I_0 \subsetneq I_1 \subsetneq ..., \tag{1}$$

showing that R fails the ascending chain condition and is not Noetherian.

□