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4. Show that $J(R)$ and $\mathfrak{N}(R)$ can be characterized in the following ways.

(a) $J(R)$ is the largest ideal $J \lhd R$ such that all covers below J in Ideal(R) are of abelian type. (That is, $I \prec K \leq J$ implies $K^2 \subseteq I$)

(b) $\mathfrak{N}(R)$ is the largest ideal $I \lhd R$ such that there is a well-ordered chain of ideals

$$0 = I_0 \leq I_1 \leq I_2 \leq \cdots \leq I_\mu = I$$

such that

(i) $I_{\alpha+1}$ is abelian over I_α for all α, and

(ii) $I_\lambda = \bigcup_{\kappa<\lambda} I_\kappa$ whenever λ is a limit ordinal.

Proof.
(a) Let $I \prec K \leq J(R)$. Viewing I, K as R-modules, we have that K/I is a simple module. Therefore, $J(R)K \subseteq I$. This inclusion still holds if we shrink $J(R)K$ to K^2. Therefore, all covers below $J(R)$ are of abelian type.

We now need to show that $J(R)$ is the *largest* ideal with this property. That is, if L has the property that all covers below it are of abelian type, then $L \leq J(R)$.

For contradiction assume that $L \not\leq J(R)$, then, there exists a maximal ideal $\mathcal{M}$ such that $L \not\leq \mathcal{M}$. Therefore, we have that $L + \mathcal{M} = R$ and we have perspective intervals: $L \cap \mathcal{M}$ to L and $\mathcal{M}$ to R. But $\mathcal{M}$ is maximal, therefore, $L \cap \mathcal{M} \prec L$. By L's covering property, $L^2 \leq L \cap \mathcal{M}$ and thus $L^2 \leq \mathcal{M}$. But, since $\mathcal{M}$ is maximal, it's prime, and hence, $L \leq \mathcal{M}$, which is a contradiction.

□

(b) Let I be an ideal with a well ordered chain satisfying (i) and (ii), we need to show that all the elements of I are nilpotent. Certainly, all the elements of the 0 ideal are nilpotent. If all the elements of I_α are nilpotent, then, because $I_{\alpha+1}^2 \leq I_\alpha$ we have that $a \in I_{\alpha+1}$ satisfies $a^2 \in I_\alpha$ and is hence nilpotent. Lastly, let λ be a limit ordinal and assume that I_κ is composed of nilpotent elements for all $\kappa < \lambda$. By property (ii), each element $a \in I_\lambda$ is already in I_κ for some $\kappa < \lambda$ and is hence nilpotent. This shows that all ideals with a well ordered chain satisfying (i),(ii) are in $\mathfrak{N}(R)$ and this particularly applies to the *largest* ideal with these properties.

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Conversely, we can express $\mathfrak{N}(R)$ with a well-order chain in the following way:

Let $I_0 := (0)$. If possible, choose an element $x_1 \in \mathfrak{N}(R) \setminus I_0$ such that $x_1^2 \in I_0$, that is, choose an x_1 such that $x_1^2 = 0$, and let $I_1 := \langle x_1, I_0 \rangle = (x_1)$. We can generalize this to a successor ordinal $\kappa + 1$ by finding (if possible) $x_{\kappa+1} \in \mathfrak{N}(R) \backslash I_\kappa$ such that $x_{\kappa+1}^2 \in I_\kappa$ and defining $I_{\kappa+1} = \langle x_{\kappa+1}, I_\kappa \rangle$. And for a limit ordinal λ, we define $I_\lambda = \bigcup_{\kappa<\lambda} I_\kappa$.

By construction, for $\kappa \leq \gamma$ we have $I_\kappa \leq I_\gamma$, and since we're indexing by ordinals, this is a well ordered chain. Because all the generators are from $\mathfrak{N}(R)$, this is a chain of ideals in $\mathfrak{N}(R)$.

An element of $I_{\kappa+1}^2$ has the form $a_1 b_1 + ... + a_n b_n$ for some n and $a_i, b_i \in \{x_{\kappa+1}\} \cup I_{\kappa+1}$. If $a_i = b_i = x_{\kappa+1}$ then $a_i b_i = x_{\kappa+1} \in I_\kappa$, and if one of a_i, b_i is in I_κ then the product is immediately in I_κ. Therefore, $I_{\kappa+1}^2 \leq I_\kappa$ satisfying (i). Property (ii) follows by our construction for I_λ for limit ordinals.

This chain terminates at some ordinal μ. This is because at each (successor) ordinal, we're adjoining one element of $\mathfrak{N}(R)$ and there's a set-many elements in $\mathfrak{N}(R)$ while the collection of ordinals forms a proper class.

This shows that $I_\mu \leq \mathfrak{N}(R)$. And since this construction terminates, no more elements could be adjoined, meaning that for all $\alpha \in \mathfrak{N}(R) \setminus I_\mu$ we have $\alpha^2 \notin I_\mu$. We now need to show that $\mathfrak{N}(R) \leq I_\mu$.

For contradiction, let $\alpha \in \mathfrak{N}(R) \backslash I_\mu$, so that α is nilpotent but $\alpha^2 \notin I_\mu$. There exists an n such that $\alpha^{2^n} = 0 \in I_\mu$. $\alpha^{2^{n-1}}$ is nilpotent and it squares to 0 which is in I_μ, therefore, $\alpha^{2^{n-1}} \in I_\mu$. We can continue this process proving that $\alpha^{2^{n-2}}, \alpha^{2^{n-3}}, ...$ are in I_μ, but eventually, we'll find some $\alpha^{2^{s-1}} \notin I_\mu$ with $\alpha^{2^s} \in I_\mu$, which contradicts the fact that I_μ contains all the nilpotent elements that square into it.

$\square$