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3. Prove that the Jacobson radical contains no nonzero idempotents in each of the following ways:

(a) using the characterization of $J(R)$ as the intersection of maximal ideals.

(b) using the characterization of $J(R)$ as the largest ideal J such that $1 + J$ consists of units.

(c) using the characterization of $J(R)$ as the intersection of annihilators of all simple modules.

Solution: We first summarize two calculations in a small lemma that is useful in parts (a) and (b):

Lemma: Let e be an idempotent of R. Then $1 - e$ is also an idempotent of R and if $1 - e$ is a unit, then $e = 0$.

Proof: First, observe that since $e^2 = e$, we have:

$$(1 - e)^2 = 1 - 2e + e^2 = 1 - 2e + e = 1 - e$$

so $1 - e$ is also an idempotent of R. Now suppose that $1 - e$ is a unit. Then letting r denote the multiplicative inverse of $1 - e$, we can perform the following calculation:

$$e = 1 \cdot e = (r \cdot (1 - e)) \cdot e = r \cdot ((1 - e) \cdot e) = r \cdot (e - e^2) = r \cdot (e - e) = r \cdot 0 = 0$$

$\square$

Suppose that e is an idempotent contained in $J(R)$.

(a) Suppose for sake of contradiction that M is a maximal ideal containing $1 - e$. M also contains e since $J(R)$ is the intersection of all maximal ideals of R and $e \in J(R)$. Then $1 = (1 - e) + e \in M$, so $M = R$, contradicting the maximality of M. Hence, no maximal ideal of R contains $1 - e$, which means that $1 - e$ is a unit. By the lemma, $e = 0$, so $J(R)$ contains no nonzero idempotents.

(b) Since $J(R)$ is an ideal, $-e \in J(R)$ and so $1 - e \in 1 + J(R)$ is a unit. By the lemma, $e = 0$, so $J(R)$ contains no nonzero idempotents.

(c) Suppose for sake of contradiction that e is nonzero. Let $(P, \subseteq)$ be the poset of ideals of R which do not contain the ideal (e).

Claim: $(P, \subseteq)$ has a maximal element.

Proof: Since e is nonzero, the ideal $\{0\}$ does not contain (e), and so P is nonempty.

Next, let $\{I_\alpha\}_{\alpha<\beta}$ be a chain of ideals in $(P, \subseteq)$ where β is some nonzero ordinal. We will show that $I := \bigcup_{\alpha<\beta} I_\alpha$ belongs to P. First, to see that I is an ideal, let $x, y \in I$. Then there exist ordinals $\alpha_1 \leq \alpha_2$ such that $x \in I_{\alpha_1}$ and $y \in I_{\alpha_2}$. But $I_{\alpha_1} \subseteq I_{\alpha_2}$ so $x + y \in I_{\alpha_2} \subseteq I$. Additionally,

$$RI = R \bigcup_{\alpha<\beta} I_\alpha = \bigcup_{\alpha<\beta} RI_\alpha \subseteq \bigcup_{\alpha<\beta} I_\alpha$$

since each I_α is an ideal. Therefore, I is an ideal. Suppose for sake of contradiction that $(e) \subseteq I$. Then $e \in I$, so there is some $\alpha < \beta$ such that $e \in I_\alpha$. But then $(e) \subseteq I_\alpha$, which contradicts the fact that $I_\alpha \in P$. Hence, $I \in P$. Moreover, I is an upper bound for $\{I_\alpha\}_{\alpha<\beta}$ by definition, and so $(P, \subseteq)$ is inductively ordered. By Zorn's Lemma, there is a maximal element of $(P, \subseteq)$. □

Let M be a maximal element of $(P, \subseteq)$. We now show that $(M + (e))/M$ is a simple R-module under the action $r \cdot (s + M) = rs + M$. First, observe that if N is a submodule of $M + (e)$ under the action of multiplication, then N is an abelian group and for all $r \in R$ and $n \in N$, $rn \in N$. In other words, every submodule of $M + (e)$ is an ideal of R. By the lattice isomorphism theorem, in order to show $(M + (e))/M$ is simple, we need only show that there are no ideals of R strictly between M and $M + (e)$. Let I be an ideal of R and suppose $M \subseteq I \subseteq M + (e)$. Also suppose that $I \neq M$. Since M is maximal for the property of not containing (e), I contains (e). I also contains M, so I contains $M + (e)$, the least ideal containing both M and (e). Hence, $I = M + (e)$ and there are no ideals strictly between M and $M + (e)$, and so $(M + (e))/M$ is a simple R-module.

Finally, observe that $e \cdot (e + M) = e^2 + M = e + M = 0 + M$ since e belongs to $J(R)$, the intersection of annihilators of simple R-modules. This implies that $e \in M$, but by definition, M does not contain e! This is a contradiction, so $e = 0$.