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2.(Nilradical versus Jacobson radical)

(a) Show that $\mathfrak{N}(R \times S) = \mathfrak{N}(R) \times \mathfrak{N}(S)$ and $J(R \times S) = J(R) \times J(S)$. Hence, if the nilradical and the Jacobson radical are equal in each coordinate of a product, then they are equal in the product.

(b) Show the result of part (a) does not hold for infinite products by showing that the nilradical and Jacobson radical are equal in all coordinates of the product $T = \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8 \times \cdots$, but $\mathfrak{N}(T) \neq J(T)$.

Proof.

(a) Suppose that $(r, s) \in \mathfrak{N}(R \times S)$. Then for some positive integer m, $(r, s)^m = 0$. Parsing this out, we get $(r, s)^m = (r^m, s^m) = (0_R, 0_S)$, so we have $r^m = 0_R$ and $s^m = 0_S$. This means $r \in \mathfrak{N}(R)$ and $s \in \mathfrak{N}(S)$, so that $(r, s) \in \mathfrak{N}(R) \times \mathfrak{N}(S)$. Now suppose that $(r, s) \in \mathfrak{N}(R) \times \mathfrak{N}(S)$. Then for some positive integers m, n, $r^m = 0_R$ and $s^n = 0_S$. We have $(r, s)^{m+n} = (r^{m+n}, s^{m+n}) = (r^m r^n, s^m s^n) = (0_R r^n, s^m 0_s) = (0_R, 0_S)$ so that $(r, s) \in \mathfrak{N}(R \times S)$. We conclude that $\mathfrak{N}(R \times S) = \mathfrak{N}(R) \times \mathfrak{N}(S)$.
Suppose $(r, s) \in J(R \times S)$. That means for any $(u, v) \in R \times S$, the element $1-(u, v)(r, s)$ is invertible. Parsing this out, we get that $(1_R, 1_S) - (u, v)(r, s) = (1_R - ur, 1_S - vs)$, so we have $1_R - ur$ is invertible for any $u \in R$ and $1_S - vs$ is invertible for any $v \in S$. This means $r \in J(R)$ and $s \in J(S)$ so that $(r, s) \in J(R) \times J(S)$. Now assume we have $(r, s) \in J(R) \times J(S)$. This means that $1_R - ur$ is invertible for any $u \in R$ and $1_S - vs$ is invertible for any $v \in S$. Then $1 - (u, v)(r, s)$ is invertible for any $(u, v) \in R \times S$ so that $(r, s) \in J(R \times S)$. We conclude that $J(R \times S) = J(R) \times J(S)$. $\square$

(b) The claim is that $\mathfrak{N}(\mathbb{Z}_{2^n}) = J(\mathbb{Z}_{2^n}) = (2)$ for any $n \geq 1$.
First, $2^n = 0$, so $2 \in \mathfrak{N}(\mathbb{Z}_{2^n})$, so $(2) = \mathfrak{N}(\mathbb{Z}_{2^n})$ since (2) is maximal and the nilradical is never the whole ring with unity.
Secondly, all the ideals of $\mathbb{Z}_{2^n}$ are of the form 2^m for some integer $1 \leq m \leq n$. Then the only maximal ideal is $(2^1) = (2)$ and $(2) = J(\mathbb{Z}_{2^n})$.
Despite that, we do get that $J(T) \neq \mathfrak{N}(T)$. Take the element $j = (2, 2, 2, \ldots)$. We have that $1 - tj$ is odd for any $t \in T$. In $(\mathbb{Z}_{2^n})$, all odd numbers are invertible. Hence, $j \in J(T)$. However, assume by way of contradiction that there exists some positive integer m such that $j^m = 0$. But that implies that $2^m = 0$ in $\mathbb{Z}_{2^{m+1}}$. So we have $j \notin \mathfrak{N}(T)$ and ultimately $J(T) \neq \mathfrak{N}(T)$. $\square$