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1. Classify all the maximal subrings of $\mathbb{Q}$, and show that any two of them have homeomorphic spectra.

First, we prove a lemma:

Lemma. *If R is an integral domain, then Spec(R) is irreducible.*

Proof of Lemma: Suppose that $\operatorname{Spec}(R)$ can be written as a union of closed subsets, $\operatorname{Spec}(R) = V(I) \cup V(J)$ for some ideals $I, J \subseteq R$, and suppose that $V(J) \neq \operatorname{Spec}(R)$. We will show that $V(I) = \operatorname{Spec}(R)$. As R is an integral domain, (0) is the unique minimal prime ideal, so $V(I) = \operatorname{Spec}(R)$ if and only if $I = (0)$. Thus $J \neq (0)$, since $V(J) \neq \operatorname{Spec}(R)$. Now $V(I) \cup V(J) = V(IJ) = \operatorname{Spec}(R)$, so $IJ = (0)$. Since $J \neq (0)$, it has some nonzero element j. From $IJ = (0)$, we conclude that $ij = 0$ for all $i \in I$, so since R is an integral domain, it must be that $I = (0)$. Hence $V(I) = \operatorname{Spec}(R)$, so $\operatorname{Spec}(R)$ is irreducible. □

Proof of Problem 1: We will classify all subrings of $\mathbb{Q}$, then identify the maximal ones. Let $R \subseteq \mathbb{Q}$ be a subring. First, we must have $0, 1 \in R$ by definition of a subring (Atiyah–MacDonald), and R must be closed under addition and additive inverses, so $\mathbb{Z} \subseteq R$.

Next, define the set $S = \left\{p \in \mathbb{Z}_{\geq 0} : p \text{ prime}, \frac{1}{p} \in R\right\}$. This set is not multiplicatively closed, but we can create the multiplicative closure of S, call it $\tilde{S}$, by adding 1 and adding all possible products of elements of S. Almost by definition of S (and $\tilde{S}$), we will have $\tilde{S}^{-1}\mathbb{Z} \subseteq R$. We claim that $\tilde{S}^{-1}\mathbb{Z} = R$. Suppose, by way of contradiction, that there is some $\frac{r}{s} \in R$ such that $\frac{r}{s} \notin \tilde{S}^{-1}\mathbb{Z}$. We assume that $\frac{r}{s}$ is in lowest terms, so r and s have no common factors. Now $\frac{r}{1} \in \tilde{S}^{-1}\mathbb{Z}$, so it must be that $\frac{1}{s} \notin \tilde{S}^{-1}\mathbb{Z}$. Thus there must be some prime factor p of s such that $p \notin S$. If we write $s = p \cdot p_1^{n_1} \cdots p_m^{n_m}$, then we have

$$\frac{r}{p} = \frac{r}{s} \cdot \frac{p \cdot p_1^{n_1} \cdots p_m^{n_m}}{1} \in R.$$

By assumption, p does not divide r, so $\gcd(p, r) = 1$. Thus we can find some $u, v \in \mathbb{Z}$ such that $1 = up + vr$. Dividing both sides by p, we get $\frac{1}{p} = u + v\frac{r}{p} \in R$, where the inclusion follows since R is a ring (and $\mathbb{Z} \subseteq R$). This is a contradiction, as it would imply that $p \in S$. Therefore, we must have $\tilde{S}^{-1}\mathbb{Z} = R$.

Thus all subrings of $\mathbb{Q}$ look like $\tilde{S}^{-1}\mathbb{Z}$ for some set of primes S (as above). Now a maximal such subring (which is not all of $\mathbb{Q}$) would arise when S is missing a single prime (the larger S is, the larger $\tilde{S}^{-1}\mathbb{Z}$ will be), i.e., $S = \{p \in \mathbb{Z}_{\geq 0} : p \text{ prime}\} \setminus \{q\}$ for a prime q. In this case, $\tilde{S} = \mathbb{Z} \setminus (q)$, and we denote $\tilde{S}^{-1}\mathbb{Z} = \mathbb{Z}_{(q)}$. Hence the maximal subrings of $\mathbb{Q}$ are those of the form $\mathbb{Z}_{(p)}$ for some prime p.

Finally, we show that the spaces $\operatorname{Spec}(\mathbb{Z}_{(p)})$ are homeomorphic for all primes p. We know that prime ideals in $\mathbb{Z}_{(p)}$ are in bijection with prime ideals of $\mathbb{Z}$ that do not intersect $\mathbb{Z} \setminus (p)$, i.e., prime ideals contained in (p). Thus $\operatorname{Spec}(\mathbb{Z}_{(p)})$ will have two points: one corresponding to the prime ideal (0) and one corresponding to the prime ideal (p), and we will write $\operatorname{Spec}(\mathbb{Z}_{(p)}) = \{(0), (p)\}$. Now $\{(p)\}$ is the vanishing locus of the ideal (p) in $\mathbb{Z}_{(p)}$, so $\{(p)\} \subseteq \operatorname{Spec}(\mathbb{Z}_{(p)})$ is closed. $\mathbb{Z}_{(p)}$ is an integral domain, so by the Lemma, $\operatorname{Spec}(\mathbb{Z}_{(p)})$ is irreducible (so also connected). Therefore, $\{(0)\}$ cannot be closed. Thus the open sets of

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$\mathrm{Spec}(\mathbb{Z}_{(p)})$ are $\{\varnothing, \{(0)\}, \{(0), (p)\}\}$. Therefore, for any prime p, $\mathrm{Spec}(\mathbb{Z}_{(p)})$ is homeomorphic to the two point space $\{a, b\}$ with open sets $\{\varnothing, \{a\}, \{a, b\}\}$. Hence all of the maximal subrings have homeomorphic spectra. $\square$