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9. Show that the map $\mathbf{Idl}(R) \to L : I \mapsto \mathbf{nil}(I)$ from the lattice of ideals of R to the lattice of semiprime ideals is a homomorphism with respect to binary $\wedge$ and infinitary $\bigvee$.

Proof: Let I and J be ideals of a commutative ring R and let φ denote the map of interest. We first show that $\varphi(I \wedge J) = \varphi(I) \wedge \varphi(J)$. Recall that in both $\mathrm{Idl}(R)$ and L, the meet of two ideals is their intersection. Let $r \in \varphi(I \wedge J)$. Then there is some $n \in \mathbb{N}$ such that $r^n \in I \wedge J = I \cap J$. For this same n, $r^n \in I$ and $r^n \in J$, and so $r \in \varphi(I) \cap \varphi(J) = \varphi(I) \wedge \varphi(J)$. Thus, $\varphi(I \wedge J) \subseteq \varphi(I) \wedge \varphi(J)$. Now let $r \in \varphi(I) \wedge \varphi(J)$. Then there are values $n_1, n_2 \in \mathbb{N}$ such that $r^{n_1} \in I$ and $r^{n_2} \in J$. Since I is an ideal,

$$r^{\max(n_1,n_2)} = r^{\max(n_1,n_2)-n_1} \cdot r^{n_1} \in I.$$

Note that $\max(n_1, n_2) - n_1$ is non-negative, so this expression is well-defined. Through a similar argument, we see that $r^{\max(n_1,n_2)}$ belongs to J as well. Hence, $r \in \varphi(I \cap J) = \varphi(I \wedge J)$, and so $\varphi(I) \wedge \varphi(J) \subseteq \varphi(I \wedge J)$. In conclusion, $\varphi(I) \wedge \varphi(J) = \varphi(I \wedge J)$.

Now let κ be any set and let I_k be an ideal of R for each $k \in \kappa$. We now show that $\varphi(\bigvee_{k\in\kappa} I_k) = \bigvee_{k\in\kappa} \varphi(I_k)$. Before we begin, recall that the join (in $\mathrm{Idl}(R)$) of a set of ideals is the sum of those ideals, and the join (in L) of a set of ideals is the radical of the sum of those ideals. Thus, our goal is to show:

$$\sqrt{\sum_{k\in\kappa} I_k} = \sqrt{\sum_{k\in\kappa} \sqrt{I_k}}$$

($\subseteq$) Since $\sqrt{\cdot}$ is extensive, we have that for all $k \in \kappa$, $I_k \subseteq \sqrt{I_k}$. Sums are monotone in each of their coordinates, so $\sum_{k\in\kappa} I_k \subseteq \sum_{k\in\kappa} \sqrt{I_k}$. Finally, $\sqrt{\cdot}$ is also monotone, so $\sqrt{\sum_{k\in\kappa} I_k} \subseteq \sqrt{\sum_{k\in\kappa} \sqrt{I_k}}$.

($\supseteq$) Let $r \in \sum_{k\in\kappa} \varphi(I_k)$. By the definition of sum, there exist finitely many indices $k_1, ..., k_s \in \kappa$ and elements $r_1 \in \varphi(I_{k_1})$, ..., $r_s \in \varphi(I_{k_s})$ such that $r = r_1 + \cdots + r_s$. Moreover, there are values $n_1, ..., n_s$ such that $r_i^{n_i} \in I_{k_i}$ for $i \in \{1, ..., s\}$. Since R is commutative, we may use the multinomial theorem to calculate powers of r. Specifically, for $n \in \mathbb{N}$, we have

$$\begin{aligned} r^n &= (r_1 + \cdots + r_s)^n \\ &= \sum_{e_1+\cdots+e_s=n} \binom{n}{e_1, ..., e_s} \prod_{i=1}^{s} r_i^{e_i} \end{aligned}$$

We intend to show that for some value of n, $r^n \in \bigvee_{k\in\kappa} I_k$. For this purpose, we need only choose a sufficiently high value of n so that $e_i \geq n_i$ for some value of i in each term of the sum above. This is sufficient because then $r_i^{e_i} \in I_{k_i}$ for some i in each term of the sum, but I_{k_i} is an ideal, so the entire term will belong to I_{k_i}. Ultimately, r^n will be a finite sum of elements of I_{k_1}, ..., I_{k_s}, and so it will belong to $\sum_{k\in\kappa} I_k$.

Let $n = s \cdot \max(n_1, ..., n_s)$ and choose any exponents $e_1, ..., e_s$ satisfying $e_1 + \cdots + e_s = n$. Then for all $i \in \{1, ..., s\}$, we have that $n/s = \max(n_1, ..., n_s) \geq n_i$. Additionally, $e_i \geq n/s$ for some $i \in \{1, ..., s\}$ (otherwise, we would have $n = e_1 + \cdots + e_s < n/s + \cdots + n/s = n$). Combining the last two sentences, we obtain $e_i \geq n/s = \max(n_1, ..., n_s) \geq n_i$ for some $i \in \{1, ..., s\}$. By the preceeding discussion, $r^n \in \sum_{k\in\kappa} I_k$, and so $r \in \sqrt{\sum_{k\in\kappa} I_k}$. Our choice of r was arbitrary, so $\sum_{k\in\kappa} \sqrt{I_k} \subseteq \sqrt{\sum_{k\in\kappa} I_k}$. Finally, $\sqrt{\cdot}$ is monotone and idempotent, so $\sqrt{\sum_{k\in\kappa} \sqrt{I_k}} \subseteq \sqrt{\sqrt{\sum_{k\in\kappa} I_k}} = \sqrt{\sum_{k\in\kappa} I_k}$.

$\square$