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Problem 7.

(a) Show that if $(P_i)_{i\in I}$ is a chain of primes, then $\bigcup_{i\in I} P_i$ and $\bigcap_{i\in I} P_i$ are primes.

(b) Show that if I is an ideal contained in a prime P, then there is a prime ideal P' such that $I \subseteq P' \subseteq P$ and P' is "minimal over I".

(c) Show that if I is an ideal containing a prime P, then there is a prime ideal P' such that $P \subseteq P' \subseteq I$ and P' is "maximal under I".

Theorem 1. *Let $(P_i)_{i\in I}$ be a chain of primes, then $\bigcup_{i\in I} P_i$ and $\bigcap_{i\in I} P_i$ are primes.*

Proof.

$\bigcup_{i\in I} P_i$ **is prime:** We first show that $\bigcup_{i\in I} P_i$ is an ideal. Suppose $a, b \in \bigcup_{i\in I} P_i$, then there are i_0 and i_1 such that $a \in P_{i_0}$ and $b \in P_{i_1}$. The collection of P_i form a chain, so we may assume without loss of generality that $P_{i_0} \subseteq P_{i_1}$. That is, $a, b \in P_{i_1}$. Since P_{i_1} is an ideal then $a + b \in P_{i_1} \subseteq \bigcup_{i\in I} P_i$ and $ra \in P_{i_1} \subseteq \bigcup_{i\in I} P_i$ for all $r \in R$.

If $ab \in \bigcup_{i\in I} P_i$, then there exists i_0 such that $ab \in P_{i_0}$. Since P_{i_0} is a prime, then either a or b is in P_{i_0}, so a or b is in $\bigcup_{i\in I} P_i$.

$\bigcap_{i\in I} P_i$ **is prime:** Suppose $a, b \in \bigcap_{i\in I} P_i$, then for all $i \in I$, $a, b \in P_i$. Each P_i is an ideal, so $a + b \in P_i$ and $ra \in P_i$ for all $r \in R$ and $i \in I$. It follows that $a + b \in \bigcap_{i\in I} P_i$ and $ra \in \bigcap_{i\in I} P_i$.

Suppose $ab \in \bigcap_{i\in I} P_i$. If a and b are in P_i for all i, then a and b will also be $\bigcap_{i\in I} P_i$. If not, then without loss of generality suppose there exists i_0 such that $b \notin P_{i_0}$. Then, $a \in P_{i_0}$ and clearly $a \in P_{i_1}$ for any $i_1 \in I$ such that $P_{i_0} \subseteq P_{i_1}$. On the other hand, if $P_{i_1} \subseteq P_{i_0}$, suppose for contradiction that $a \notin P_{i_1}$. Then $b \in P_{i_1}$ because P_{i_1} is prime, but this contradicts that $P_{i_1} \subseteq P_{i_0}$. It follows that $a \in P_i$ for all $i \in I$ so $a \in \bigcap_{i\in I} P_i$.

□

Definition 1. *Let I be an ideal. A prime P' containing I is called* ***minimal over*** *I if there does not exists another prime P containing I such that $I \subseteq P \subsetneq P'$. A prime P' contained in I is called* ***maximal under*** *I if there does not exist another prime P contained within I satisfying $P' \subsetneq P \subseteq I$.*

Corollary 1. *Let I be an ideal and P be a prime containing I. Then there exists a prime P' contained in P that is minimal over I.*

Proof. Consider $\mathcal{C} = \{Q \text{ prime ideal} : I \subseteq Q \subseteq P\}$, with the partial ordering $P \preceq Q$ if $Q \subseteq P$. Observe that $\mathcal{C}$ is nonempty because $P \in \mathcal{C}$. Let $\{P_i\}_{i\in I}$ be a chain, and let $\tilde{P} = \bigcap_{i\in I} P_i$. Clearly, $I \subseteq \tilde{P} \subseteq P_i \subseteq P$ for all i. Then by Proposition 1 $\tilde{P}$ is prime. That is, $\tilde{P}$ is an upper bound for this chain. Zorn's lemma then guarantees a maximal element P' with respect to this ordering. That is, there are no prime Q satisfies $I \subseteq Q \subsetneq P'$. □

Corollary 2. *Let I be an ideal and P be a prime contained in I. Then there exists a prime P' containing P that is maximal under I.*

Proof. Define $\mathcal{C} = \{Q \text{ prime ideal} : P \subseteq Q \subseteq I\}$ with the ordering $P \preceq Q$ if $P \subseteq Q$. Again, $\mathcal{C}$ is nonempty because $P \in \mathcal{C}$. For any chain $\{P_i\}_{i \in I}$ define $\tilde{P} = \bigcup_{i \in I} P_i$. Since $P \subseteq P_i \subseteq I$ for all i, then $P \subseteq \tilde{P} \subseteq I$. Furthermore, $\tilde{P}$ is prime by Proposition 1 so $\tilde{P}$ is an upper bound for this chain. By Zorn's lemma, the is a prime P' maximal with respect to this ordering. That is, there are no primes Q which satisfies $P' \subsetneq Q \subseteq I$. $\square$