Connor Meredith
Mateo Muro
Adrian Neff

6. Let I be a minimal nonzero ideal of the commutative ring R.

(a) Show that $(0 : I)$ is a maximal ideal.

(b) Show that if $I^2 = I$, then K is a complement to I and $R \cong R/I \times R/K$.

Proof.

(a) For brevity, let $K = (0 : I)$. We consider two cases. I^2 is an ideal contained in I, so it is either equal to I itself or is equal to the zero ideal.
Consider the case where I squares to itself. Choose nonzero $a \in I$. $(a) \subseteq I$ is nonzero so it must equal I. Since $(a^2) = (a)^2 = I^2 = I$, it follows that a^2 also generates $I = (a)$ and therefore there must be some $r \in R$ such that $ra^2 = a$. We know $r \notin K$ for otherwise $r(a^2) = 0 \neq (a)$. In particular, we have $ra^2 \neq 0$. Multiplying by r yields $r^2a^2 = ra$ and $ra \neq 0$ for otherwise $r \in K$. Let $e = ra$ and write $I = (e)$. Notice that $e^2 = rara = r^2a^2 = ra = e$. So e is an idempotent element. If $e(1 - e) = e - e^2 = e - e = 0$, then $(1 - e)$ annhilates e. This shows that the ideal $(1 - e)$ is contained in K. The ideals (e) and $(1 - e)$ are complementary (the sum of generators generates R). Since $(e) = I$ is a minimal ideal of R, and the complement of a minimal element of a modular lattice is a maximal element, it follows that $(1 - e)$ is a maximal ideal of R. Since K is an ideal containing a maximal ideal, and $1 \notin K$ $(1I = I)$, it follows from maximality that $K = (1 - e)$, which is a maximal ideal.
Consider the case where $I^2 = (0)$. By minimality, I is generated by any one of its nonzero elements. For if $a \in I$, then $(a) \subseteq I$ and since a is nonzero, $(a) = I$. We know by applying the First Isomorphism Theorem for Modules to the R-module homomorphism $\phi : R \mapsto (a)$, $\phi(r) = ra$ that $R/\operatorname{Ann}_R(a) \cong (a)$. Then we have $R/K \cong I$ as R modules. Since (a) is a minimal ideal, it is simple as an R-module. Therefore R/K is simple, so by the Correspondence Theorem K is a maximal ideal of R. □

(b) We have already shown that if $I^2 = I$ we have that K is a complement to K. By the Chinese Remainder Theorem, $R \cong R/(0) \cong R/I \times R/K$. See also the solution to calg1p3(d).

□