Commutative Algebra HW1p5

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Suppose that $I \leq R$ is a nil ideal (meaning: every element of I is nilpotent).

(a) Show that $a + I$ is a unit in R/I if and only if a is a unit in R.

(b) Show that $a + I$ is idempotent in R/I if and only if there exists an idempotent $e \in R$ such that $e + I = a + I$.

Proof.

(a) To show the if direction, suppose that a is a unit in R. Let a^{-1} be its inverse. Then

$$(a + I)(a^{-1} + I) = (a^{-1} + I)(a + I) = 1 + I.$$

To prove the converse, suppose $a + I$ is a unit. Let $b \in R$ be a representative of $(a + I)^{-1}$. Then

$$ab + I = (a + I)(b + I) = 1 + I,$$

showing $1 - ab \in I$. Since I is a nil ideal, there must be some $n \in \mathbb{N}$ such that $(1 - ab)^n = 0$. Expanding this out,

$$0 = 1 - \binom{n}{1}ab + ... + (-1)^n(ab)^n = 1 - a[\binom{n}{1}b + ... + (-1)^n a^{n-1}\binom{n}{n}b^n].$$

Since R is commutative, this shows that a has an inverse given by $\binom{n}{1}b + ... + (-1)^n\binom{n}{n}a^{n-1}b^n$.

(b) To show the if direction, suppose there exists an idempotent $e \in R$ such that $e + I = a + I$. Then

$$(a + I)^2 = (e + I)^2 = e^2 + I = e + I = a + I.$$

To show the only if direction, suppose that the coset $a + I$ is idempotent. Then $a + I = (a + I)^2 = a^2 + I$, showing that $a(a - 1) = a^2 - a \in I$. Since I is a nil ideal, there must be some $n \in \mathbb{N}$ such that

$$[a(a - 1)]^n = 0 \tag{†}$$

.

Expanding the equation $1 = (a + (1 - a))^{2n}$ and using nilpotence we get

$$\begin{aligned}1 =& \binom{2n}{0}a^{2n} + \binom{2n}{1}a^{2n-1}(1 - a) + ... + \binom{2n}{n}a^n(1 - a)^n + ... + (1 - a)^{2n} \\ =& a^n\left[\binom{2n}{0}a^n + ... + \binom{2n}{n-1}a(1 - a)^{n-1}\right] + (1 - a)^n\left[\binom{2n}{n+1}a^{n-1}(1 - a) + ... + \binom{2n}{2n}(1 - a)^n\right].\end{aligned} \tag{‡}$$

If we let $x = a^n\left[\binom{2n}{0}a^n + ... + \binom{2n}{n-1}a(1-a)^{n-1}\right]$ and $y = (1-a)^n\left[\binom{2n}{n+1}a^{n-1}(1-a) + ... + \binom{2n}{2n}(1-a)^n\right]$ then by ‡ we see that $x + y = 1$. Also, by (†), it follows that $xy = 0$. That makes x and y complementary idempotents. The idempotence of x, for example, follows from

$$x = x(x + y) = x^2 + xy = x^2.$$

To show that some element in $a + I$ is idempotent, it suffices to show that $x + I = a + I$. Because $a(a - 1) \in I$, dropping all multiples of $a(a - 1)$ from x will give an element in the same coset modulo I. Dropping these terms and using the idempotence of $a + I$ we get

$$x + I = a^{2n} + I = (a + I)^{2n} = a + I.$$

□