COMMUTATIVE ALGEBRA HOMEWORK 1

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Problem (4)**.** Show that the ideals of $R \times S$ are of the form $I \times J$ where $I \triangleleft R$ and $J \triangleleft S$. Show that the prime (maximal) ideals have the form $P \times S$ and $R \times Q$ for prime (maximal) ideals $P \triangleleft R$ and $Q \triangleleft S$.

Lemma. *If $f : R \to S$ is surjective, then f takes ideals of R to ideals of S.*

Proof. It's clear that $f(I)$ is an additive subgroup, and surjectivity gives $Sf(I) = f(R)f(I) = f(RI) = f(I)$ so $f(I)$ is an ideal. $\square$

Claim. Ideals of $R \times S$ are products of ideals of R and S.

Proof. It's clear that products of ideals are ideals of the product. Conversely, let K be an ideal of $R \times S$ and π_R, π_S denote the canonical projections. We have by the lemma that $\pi_R(K)$ and $\pi_S(K)$ are ideals of R and S, respectively. Of course $K \subseteq \pi_R(K) \times \pi_S(K)$. For the other direction, if $(r, s) \in \pi_R(K) \times \pi_S(K)$ then r is the image of some $(r, s') \in K$ and s is the image of some $(r', s) \in K$. Then $(r, s')(1, 0) + (r', s)(0, 1) = (r, s)$ must be in K as well. Hence $K = \pi_R(K) \times \pi_S(K)$. $\square$

Claim. The maximal ideals of $R \times S$ are either $P \times S$ for maximal ideals P of R or $R \times Q$ for maximal ideals Q of S.

Proof. Again it's straightforward that $P \times R$ for maximal $P < R$ is maximal in $R \times S$ (the only proper ideals above it must be $P' \times R$ for $P < P' < R$, contradicting the maximality of P), and similarly for $R \times Q$

Conversely, let $K = I \times J$ be a proper ideal of $R \times S$. If both $I < R$ and $J < S$, then $R \times J$ and $I \times S$ are strictly between K and $R \times S$, and K is not maximal. So suppose without loss of generality that $J = S$ but I is not maximal in R, say $I < I' < R$, then $K < I' \times S < R \times S$, and again K is not maximal. Hence if K is maximal it must have one of the two forms described. $\square$

Claim. A prime ideal of $R \times S$ is either $P \times S$ for a prime ideal P of R or $R \times Q$ for a prime ideal Q of S.

Proof. Consider the ideal $K = P \times S$ for P prime in R and suppose $(a, b)(a', b') \in P \times S$. Then in particular $aa' \in P$ and either $a \in P$ (and $(a, b) \in K$) or $a' \in P$ (and $(a', b') \in K$). So ideals of the form described are prime.

Conversely, suppose $K = I \times J$ is prime. If both $I < R$ and $J < S$, then in particular neither of them contain 1. Then take $a \in I$, $b \in J$, and note that $(a, 1) \notin K$ and $(1, b) \notin K$ but $(a, 1)(1, b) = (a, b) \in K$, and K is not prime. So suppose without loss of generality that $J = S$ but I is not prime, then we have an $ab \in I$ with $a, b \notin I$. Hence $(a, 1)(b, 1) = (ab, 1) \in K$ with neither factor in K, and K is not prime. Hence any prime ideal must have one of the two forms described. $\square$