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3.

(a) Show that a ring R is directly decomposable as a ring if and only if it is directly decomposable when considered as an R-module. (Directly decomposable means a product of two nonzero factors.)

(b) Show that an R-module M is directly decomposable if and only if it has an idempotent endomorphism $\varepsilon : M \to M$ such that $\ker(\varepsilon) \neq 0 \neq \operatorname{im}(\varepsilon)$.

(c) Show that the R-module endomorphisms of ${}_R R$ all have the form $\varepsilon(x) = rx$ for some $r \in R$.

(d) Show that any direct decomposition of R has the form $R \cong R/(e) \times R/(1-e)$ for some idempotent $e \in R$.

Proof:

(a) ($\Rightarrow$) Suppose that R is directly decomposable as a ring, i.e., R is a direct product of nonzero rings $R \cong R_1 \times R_2$. Let $I_1 = R_1 \times 0$ and $I_2 = 0 \times R_2$. We claim that each I_i is an ideal of R, and that this is actually a direct decomposition of R as an R-module.

We show that I_1 is an ideal, and the proof for I_2 is exactly the same. First, clearly $(0,0) \in I_1$. The proof that I_1 is an abelian group under addition follows from the fact that addition in a direct product is componentwise, and R_1 is an abelian group under addition (as it is a ring). Now let $(r_1, r_2) \in R$ and $(i_1, 0) \in I_1$. Then $(r_1, r_2)(i_1, 0) = (r_1 i_1, 0) \in I_1$. Thus I_1 is an ideal.

Now since I_1 and I_2 are ideals, which are submodules of R (viewed as an R-module), we see that $R \cong I_1 \oplus I_2$ is a decomposition of R as an R-module.

($\Leftarrow$) Suppose that R is directly decomposable as an R-module, $R \cong I_1 \oplus I_2$, where $I_1, I_2 \subseteq R$ are ideals (submodules). We have

$$I_1 \cong I_1 \oplus 0 \cong (I_1 \oplus I_2)/(0 \oplus I_2) \cong R/(0 \oplus I_2) \cong R/I_2$$

$$I_2 \cong 0 \oplus I_2 \cong (I_1 \oplus I_2)/(I_1 \oplus 0) \cong R/(I_1 \oplus 0) \cong R/I_1,$$

so we see that $R \cong R/I_2 \times R/I_1$ is a decomposition of R as a ring.

(b) ($\Rightarrow$) Suppose $M \cong M_1 \oplus M_2$, with $M_i \neq 0$ for each i. The category of R-modules is abelian, so finite products have the structure of a biproduct. Thus M will come with canonical projections and inclusions, $p_j : M \to M_j$ and $\iota_j : M_j \to M$. Define $\varepsilon : M \to M$ by $\varepsilon = \iota_1 \circ p_1$. Specifically, ε acts by

$$\varepsilon(m_1, m_2) = \iota_1(p_1(m_1, m_2)) = \iota_1(m_1) = (m_1, 0).$$

First, $\ker(\varepsilon) \neq 0$, since any $(0, m_2)$ will be in the kernel, so we can take $m_2 \neq 0$. Next, $\operatorname{im}(\varepsilon) \neq 0$, since any nonzero m_1 will give a nonzero element in the image: $\varepsilon(m_1, m_2) = (m_1, 0) \neq 0$.

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Finally, we show that ε is idempotent. We check this directly:

$$\varepsilon(\varepsilon(m_1, m_2)) = \varepsilon(m_1, 0) = (m_1, 0) = \varepsilon(m_1, m_2).$$

Hence $\varepsilon \circ \varepsilon = \varepsilon$, so ε is idempotent.

($\Leftarrow$) Given such an $\varepsilon : M \to M$, we first note that $1 - \varepsilon : M \to M$ is also an endomorphism (where $1 - \varepsilon$ means $\mathrm{id}_M - \varepsilon$). Now $\mathrm{im}(\varepsilon) \neq 0$ by assumption. On the other hand, if $\mathrm{im}(1 - \varepsilon) = 0$, then we must have $\varepsilon = \mathrm{id}_M$, but this is impossible, since $\ker(\varepsilon) \neq 0$. Thus $\mathrm{im}(1 - \varepsilon) \neq 0$. We show that $M \cong \mathrm{im}(\varepsilon) \oplus \mathrm{im}(1 - \varepsilon)$ (note that this is the same as a kernel-image decomposition, $M \cong \mathrm{im}(\varepsilon) \oplus \ker(\varepsilon)$).

Let $m \in M$. Then we have $m = \varepsilon(m) + (1-\varepsilon)(m)$, so every element of M can be written as a sum of elements from $\mathrm{im}(\varepsilon)$ and $\mathrm{im}(1-\varepsilon)$. Now we show that $\mathrm{im}(\varepsilon) \cap \mathrm{im}(1-\varepsilon) = 0$, which will complete the proof (this is a characterization of direct sums). Suppose $m \in \mathrm{im}(\varepsilon) \cap \mathrm{im}(1 - \varepsilon)$. Then there are some $n, n' \in M$ such that

$$m = \varepsilon(n) = (1 - \varepsilon)(n').$$

Since ε is idempotent, we see that

$$m = \varepsilon(n) = \varepsilon(\varepsilon(n)) = \varepsilon((1 - \varepsilon)(n')) = \varepsilon(n' - \varepsilon(n')) = \varepsilon(n') - \varepsilon(n') = 0,$$

so $m = 0$, as desired. Hence $M \cong \mathrm{im}(\varepsilon) \oplus \mathrm{im}(1 - \varepsilon)$.

(c) First, we show that such a map is actually an endomorphism of ${}_RR$. Let $x, y \in R$. Then

$$\varepsilon(x + y) = r(x + y) = rx + ry = \varepsilon(x) + \varepsilon(y)$$
$$\varepsilon(xy) = r(xy) = x(ry) = x\varepsilon(y)$$

so the map is an endomorphism.

Now let $\pi : R \to R$ be any endomorphism (of ${}_RR$), and let $r = \pi(1)$. Then for any $x \in R$, we have

$$\pi(x) = \pi(x \cdot 1) = x\pi(1) = xr = rx.$$

Therefore, any such endomorphism of ${}_RR$ has the desired form.

(d) From part (a), any decomposition of R as a ring is the same as a decomposition as an R-module, so we can use the results of parts (b) and (c) on decompositions of R-modules.

From part (b), any such decomposition must arise from an idempotent endomorphism $\varepsilon : R \to R$ (an R-module endomorphism). From part (c), this endomorphism must have the form $\varepsilon(x) = ex$ for some $e \in R$. This e must be idempotent, as

$$e = \varepsilon(1) = \varepsilon(\varepsilon(1)) = \varepsilon(e) = e^2.$$

From our proof of part (b), this means that $R \cong \mathrm{im}(\varepsilon) \times \mathrm{im}(1 - \varepsilon)$. From the first isomorphism theorem, we know that $\mathrm{im}(\varepsilon) \cong R/\ker(\varepsilon)$ and $\mathrm{im}(1 - \varepsilon) \cong R/\ker(1 - \varepsilon)$. Thus we need only show that $\ker(\varepsilon) = (1 - e)$ and $\ker(1 - \varepsilon) = (e)$, then we are done.

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We know that $(1-e) \subseteq \ker(\varepsilon)$, since $\varepsilon(1-e) = e(1-e) = e - e = 0$, and kernels are ideals. Now let $r \in \ker(\varepsilon)$, so $er = 0$. Then we also have $-er = 0$. Adding r to both sides, we get $r(1-e) = r - re = r - er = r$, so $r \in (1-e)$. Thus $(1-e) = \ker(\varepsilon)$.

Similarly, we know that $(e) \subseteq \ker(1-\varepsilon)$, since $(1-\varepsilon)(e) = e - \varepsilon(e) = e - e = 0$. Now let $r \in \ker(1-\varepsilon)$, so $r - er = 0$. Then we can add er to both sides to get $r = er = re$, so $r \in (e)$. Thus $(e) = \ker(1-\varepsilon)$. Hence $R \cong R/(e) \times R/(1-e)$, as desired.

$\square$