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2. Let R be an integral domain and let K be its field of fractions. Show that the following are equivalent.

(a) For every $x \in K$, either $x \in R$ or $x^{-1} \in R$.

(b) The ideal lattice of R is a chain.

Proof.
(a) $\implies$ (b)

For contradiction, assume there exists ideals I, J with $I \not\subseteq J$ and $J \not\subseteq I$. This means that there exists $a \in I$ and $b \in J$ such that $a \notin J$ and $b \notin I$. Since K is the field of fractions, we have that $\frac{a}{b}, \frac{b}{a} \in K$, and by hypothesis (a), this means that either $\frac{a}{b} \in R$ or $\frac{b}{a} \in R$. Without loss of generality, assume that it's actually $\frac{a}{b} \in R$. Since J is an ideal, $\frac{a}{b} \cdot b \in J \implies a \in J$ and we have a contradiction.

(b) $\implies$ (a)

We want to show that for any $a, b \in R$, we have $\frac{a}{b} \in R$ or $\frac{b}{a} \in R$. Consider the principal ideals $(a), (b)$. Since the lattice is a chain, we have $(a) \subseteq (b)$ or $(b) \subseteq (a)$, so, either $a|b$ or $b|a$ in R. But the divisibility $a|b$ in R means that there exists an element $x \in R$ such that $a = xb$, and in the field of fractions, x is identified with $\frac{a}{b}$. The same holds for $b|a$, therefore, we have $\frac{a}{b} \in R$ or $\frac{b}{a} \in R$. $\square$