Commutative Algebra- HW1

Bob Kuo, Michael Levet, Chase Meadors

Problem 1. Let k be a field. Describe the ideal lattices of the following rings; and in each case, specify which ideals are maximal or prime.

(a) $k[x]$.

(b) $k[[x]]$, the ring of formal power series over k.

(c) $k((x))$, the ring of formal Laurent series over k.

Remark 1. We recall from standard Algebra texts that $k[x]$ is a PID. So the ideals of $k[x]$ are precisely of the form $(p(x))$, where $p(x) \in k[x]$. Now for $p(x), q(x) \in k[x]$, we have that $(q(x)) \subset (p(x))$ if and only if $p(x)$ divides $q(x)$.

We recall the following from the first-year Algebra sequence.

Lemma 2. Let k be a field, and consider $k[x]$. Let $p(x) \neq 0$ be an element of $k[x]$. The following are eqiuvalent.

(a) $p(x)$ is irreducible.

(b) $(p(x))$ is maximal.

(c) $(p(x))$ is prime.

Proof. We first establish the equivalence of (a) and (b). Consider the ideal $I = (p(x))$ of $k[x]$. As $p(x) \neq 0$, we have that $I \neq (0)$. We recall from standard Algebra texts that $k[x]/I$ is a field if and only if $p(x)$ is irreducible. By the characterization of maximal ideals, we have that I is maximal if and only if $k[x]/I$ is a field. So I is maximal if and only if $p(x)$ is irredicuble.

We now establish the equivalence of (a) and (c). As $k[x]$ is a PID, $k[x]$ is also a UFD. In a UFD R, we have that $f \in R$ is irreducible if and only if (f) is prime. So $p(x)$ is irreducible if and only if $(p(x))$ is prime. The result follows. $\square$

Remark 3. We note that (0) is prime, but not maximal in $k[x]$.

We next analyze the ideal lattice of $k[[x]]$, where k is a field. We begin by establishing the following criterion for identifying units in $k[[x]]$.

Lemma 4. Let R be a commutative ring with identity, and let $p(x) = \sum_{k=0}^{\infty} a_k x^k \in R[[x]]$. We have that $p(x)$ is a unit of $R[[x]]$ if and only if a_k is a unit of R.

Proof. Suppose first that $p(x)$ is a unit of $R[[x]]$, and let $q(x) = \sum_{k=0}^{\infty} b_k x^k$ be the multiplicative inverse of $p(x)$. As $p(x) \cdot q(x) = 1$, we have necessarily that $a_0 b_0 = 1$. So $b_0 = a_0^{-1} \in R$. Thus, a_0 is a unit of R.

Conversely, suppose a_0 is a unit of R. Let $q(x) = \sum_{k=0}^{\infty} b_x x^k$ be a potential multitplicative inverse of $p(x)$, where the coefficients of $q(x)$ are yet to be determined. In order for $q(x)$ to be a multiplicative inverse for $p(x)$, it is necessary and sufficient that $p(x) \cdot q(x) = 1$. We note for $i \in \mathbb{N}$, the coefficient of x^i is:

$$\sum_{j=0}^{i} a_j b_{i-j}.$$

In order for $p(x) \cdot q(x) = 1$, the coefficient of x^0 must be 1; and for $i > 0$, the coefficient of x^i must be 0. This yields the following equations:

$$\begin{aligned}
&a_0 b_0 = 1 \\
&a_0 b_1 + a_1 b_0 = 0 \\
&a_0 b_2 + a_1 b_1 + a_2 b_0 = 0 \\
&\vdots
\end{aligned}$$

Observe that as a_0 is a unit, we have that $b_0 = a_0^{-1}$. Now fix $k > 0$, we solve for b_k to obtain:

$$b_k = -a_0^{-1} \sum_{i=1}^{k} a_i b_{k-i}$$

We verify that $q(x) = \sum_{k=0}^{\infty} b^k x^k$ is indeed the inverse of $p(x)$. Observe that:

$$\begin{aligned}
p(x) \cdot q(x) &= \left(\sum_{k=0}^{\infty} a_k x^k \right) \cdot \left(\sum_{k=0}^{\infty} b_k x^k \right) \\
&= \sum_{k=0}^{\infty} x^k \left(\sum_{j=0}^{k} a_j b_{k-j} \right) \\
&= 1 + \sum_{k=1}^{\infty} x^k \left(\sum_{j=0}^{k} a_j b_{k-j} \right) \\
& \\
&= 1 + \sum_{k=1}^{\infty} x^k \left(a_0 b^k + \sum_{j=1}^{k} a_j b_{i-j} \right) \\
&= 1 + \sum_{k=1}^{\infty} x^k \left(a_0 \left(-a_0^{-1} \sum_{j=1}^{k} a_j b_{k-j} \right) + \sum_{j=1}^{k} a_j b_{k-j} \right) \\
&= 1 + \sum_{k=1}^{\infty} x^k \left(- \left(\sum_{j=1}^{k} a_j b_{k-j} \right) + \sum_{j=1}^{k} a_j b_{k-j} \right) \\
&= 1 + \sum_{k=1}^{\infty} x^k \cdot 0 \\
&= 1.
\end{aligned}$$

The result follows. □

Theorem 5. Let k be a field. The proper, non-zero ideals of $k[[x]]$ are precisely of the form (x^i) where $i \in \mathbb{N}$.

Proof. We recall from standard Algebra texts that $k[[x]]$ is a PID. Let $I = (p(x))$ be an ideal, where $p(x) = \sum_{i=0}^{\infty} a_i x^i$. Let $m \in \mathbb{N}$ be the minimum index such that $a_m \neq 0$. We claim that $I = (x^m)$. Clearly, $x^m \big| p(x)$. So $I \subset (x^m)$.

Conversely, we have that:

$$p(x) = \sum_{k=0}^{\infty} a_k x^k = x^m \sum_{k=m}^{\infty} a_k x^{k-m}.$$

By the definition of m, we have that $a_m \neq 0$ in $p(x)$. So $a_m \in k^{\times}$. Thus,

$$\sum_{k=m}^{\infty} a_k x^{k-m}$$

is a unit of $k[[x]]$. So we may write:

$$x^m = p(x) \cdot \left(\sum_{k=m}^{\infty} a_k x^{k-m} \right)^{-1} \in I.$$

It follows that $(x^m) \subset I$. The result follows. $\square$

Remark 6. We note that the ideals of $k[[x]]$ form an infinite chain with (0) as the minimum element and $k[[x]]$ as the maximum element. The proper, non-zero ideals satisfy the following condition: $(x^i) \subset (x^j)$ if and only if $i \geq j$. Furthermore, the sole maximal ideal of $k[[x]]$ is (x). In particular, as $k[[x]]$ is a PID, the maximal ideals are precisely the non-zero prime ideals. So (x) is also the sole non-zero prime ideal. For emphasis, we note that (0) is also a prime ideal of $k[[x]]$.

We finally characterize the ideal lattice of $k((x))$, where k is a field. We begin by showing that $k((x))$ is also a field here.

Theorem 7. Let k be a field. Then $k((x))$ is also a field.

Proof. Let K be the field of fractions for $k[[x]]$. We show that $K = k((x))$. We first observe that if $p(x) = \sum_{n \in \mathbb{Z}} a_n x^n \in k((x))$ is a non-zero element, we may write $p(x) = x^m \sum_{i=0}^{\infty} a_i x^i$ where $a_i \neq 0$ and $m \in \mathbb{Z}$. We note that $x^m \in K$ and $\sum_{i=0}^{\infty} a_i x^i$ is a unit of $k[[x]]$. So $p(x) \in K$. Thus, $k((x)) \subset K$. We now argue that $k((x))$ is a field. Again take $p(x) \in k((x))$ to be non-zero, and write $p(x) = x^m u$ for some $m \in \mathbb{Z}$ and some unit $u \in k[[x]]$. Observe that $(p(x))^{-1} = x^{-m} u^{-1}$. So $k((x))$ is a field. As K, the field of fractions, is the smallest field into which $k[[x]]$ can be embedded, it follows that $k((x)) = K$. $\square$

This yields the following corollary.

Corollary 8. Let k be a field. The only ideals of $k((x))$ are (0) and $k((x))$.

Remark 9. We note that (0) is the unique maximal ideal of $k((x))$, as well as the unique proper prime ideal of $k((x))$.