COMMUTATIVE ALGEBRA
HOMEWORK ASSIGNMENT IV

PROBLEMS

All rings are commutative.

1. (Toby Aldape, Bob Kuo, Connor Meredith) Let $L, N \leq M$ be A-modules. Let U be the set of primes for which $L_\mathfrak{p} \subseteq N_\mathfrak{p}$ holds. Show that U is an intersection of open sets in $\mathrm{Spec}(A)$. Show conversely that if V is any intersection of open sets in $\mathrm{Spec}(A)$, then V is exactly the set of primes for which $L_\mathfrak{p} \subseteq N_\mathfrak{p}$ holds for some submodules L, N of some module M.

2. (Ezzedine El Sai, Michael Levet, Mateo Muro) Let M be an A-module, and let $m, n \in M$ and $r \in A$ be elements.

(a) Show that the set of primes $\mathfrak{p}$ where "$m = 0$" is an open subset of $\mathrm{Spec}(A)$, and that it is all of $\mathrm{Spec}(A)$ iff $m = 0$. (Here $m \in N_\mathfrak{p}$ is shorthand for $\frac{m}{1} \in N_\mathfrak{p}$.)
(b) Same type of problem for "$m = n$".
(c) Same type of problem for "r is nilpotent".
(d) Same type of problem for "r is a unit".

A ring is ***subdirectly irreducible*** (SI) if it has a least nonzero ideal. A module is SI if it has a least nonzero submodule.

3. (Howie Jordan, Chase Meadors, Adrian Neff) Prove in each of the following ways that an SI Noetherian ring must be Artinian:

(a) Using primary decomposition: Show that A has a unique associated prime, which is a nilpotent maximal ideal. Then show that a Noetherian ring with a nilpotent maximal ideal is Artinian.
(b) Using the Krull Intersection Theorem: again, first show that A has a nilpotent maximal ideal.

4. (Toby Aldape, Bob Kuo, Connor Meredith) Let M be a f.g. SI module over a Noetherian ring A.

(a) Show that M is Artinian.

(b) Show that M has a composition series, and that all composition factors are isomorphic.

5. (Ezzedine El Sai, Michael Levet, Mateo Muro)

(a) Assume that A is Noetherian, that M is a finitely generated A-module and that $L, N \leq M$ are submodules. Show that $L \subseteq N$ iff $L_{\mathfrak{p}} \subseteq N_{\mathfrak{p}}$ holds for every $\mathfrak{p} \in \text{Ass}(M/N)$.

(b) Show that any subset $U \subseteq \text{Spec}(A)$ can be $\text{Ass}(M)$ for some A-module M. Show that any finite subset $U_0 \subseteq \text{Spec}(A)$ can be the set of associated primes of some f.g. module.

6. (Howie Jordan, Chase Meadors, Adrian Neff)

(a) Prove that if $0 \to L \to M \to N \to 0$ is exact, then $\text{Supp}(M) = \text{Supp}(L) \cup \text{Supp}(N)$.

(b) (Originally omitted hypothesis: Assume that L and N are finitely generated.) Prove that $\text{Supp}(L \otimes_A N) = \text{Supp}(L) \cap \text{Supp}(N)$.

7. (Toby Aldape, Bob Kuo, Connor Meredith)

(a) Let S be the set of elements of A that are not zero divisors. Show that S is the largest subset of A with the property that the canonical homomorphism $A \to S^{-1}A : a \mapsto a/1$ is an embedding. ($S^{-1}A$ is called the *total ring of fractions* of A.)

(b) Show that if A is Noetherian, then the total ring of fractions of A has finitely many maximal ideals. (A ring with finitely many maximal ideals is called *semilocal*.) (Hint: Consider $\text{Ass}({}_AA)$.)

8. (Ezzedine El Sai, Michael Levet, Mateo Muro) Show that if A is an integrally closed domain and $f \in A[x]$ is a monic polynomial over A, then f irreducible over A iff F is irreducible over the field of fractions of A.

9. (Howie Jordan, Chase Meadors, Adrian Neff) Suppose that $A \leq B$ is an integral extension, and that B is finitely generated as an A-algebra. Show that for every prime $\mathfrak{p} \in \text{Spec}(A)$ there are only finitely many primes of B lying over $\mathfrak{p}$.