Chain complexes, exact sequences

Chain complexes

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A *chain complex* of R-modules is a sequence

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Chain complexes of R-modules form a category where a morphism $\alpha\colon K \to L$ is an indexed family of R-linear maps such that all squares are commutative in:

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{\delta_{n+1}} & K_n & \xrightarrow{\delta_n} & K_{n-1} & \xrightarrow{\delta_{n-1}} & \cdots \\
 & & \alpha_n \downarrow & & \alpha_{n-1} \downarrow & & \\
\cdots & \xrightarrow{\varepsilon_{n+1}} & L_n & \xrightarrow{\varepsilon_n} & L_{n-1} & \xrightarrow{\varepsilon_{n-1}} & \cdots
\end{array}$$

Factoring a Complex

$$\cdots \longrightarrow K_{n+1} \xrightarrow{\delta_{n+1}} K_n \xrightarrow{\delta_n} K_{n-1} \longrightarrow \cdots$$

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$$\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 & & & & 0 & \\
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This short sequence is exact except possibly at the middle, where the homology is the same as in the original sequence.

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$p_A \iota_A = \mathrm{id}_A,\ p_A \iota_C = 0_{CA},\ p_C \iota_A = 0_{AC},\ p_C \iota_C = \mathrm{id}_C,\ \iota_A p_A + \iota_C p_C = \mathrm{id}_B.$

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For last item, write $x \in B$, as $x_{\mathrm{im}(\alpha)} + x_{\mathrm{im}(\delta)}$. Apply $\alpha\gamma + \delta\beta$ and show that x is unmoved.

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Additive Functors

$\mathcal{C}$ is *preadditive* if each hom-set $\mathrm{Hom}_{\mathcal{C}}(A,B)$ has specified abelian group operations such that $f\circ(g+h)=f\circ g+f\circ h$ and $(f+g)\circ h=f\circ h+g\circ h$. (A category is *additive* if preadditive + has finite biproducts.)

A covariant functor $F\colon \mathcal{C}\to\mathcal{D}$ between (pre)additive categories is *additive* if

$$F\colon \mathrm{Hom}_{\mathcal{C}}(A,B)\to\mathrm{Hom}_{\mathcal{D}}(F(A),F(B))$$

is an abelian group homomorphism for all $A,B\in\mathrm{Ob}(\mathcal{C})$

Examples.
The category of all R-modules is additive. (Say why!)

Representable functors between module categories are additive. (Say why!)

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But they need not preserve exact sequences.

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   For you to check: Additive functors map chain complexes to chain complexes, and split exact sequences to split exact sequences.

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Thm. Injectivity test: (Baer's Criterion) $\mathrm{Hom}(\_, Q)$ is exact on all SESs iff it is exact on those of the form $0 \to I \to R \to R/I \to 0$.

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- $\beta \otimes \mathrm{id}$ maps generators onto generators, so it is surjective.
- To show exactness at $M \otimes A$ it suffices to prove that the second map in

$$M \otimes A \xrightarrow{\nu} (M \otimes A)/I \xrightarrow{\overline{\beta \otimes \mathrm{id}}} N \otimes A$$

is invertible ($I = \mathrm{im}(\alpha \otimes \mathrm{id})$).

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- $\beta \otimes \mathrm{id}$ maps generators onto generators, so it is surjective.
- To show exactness at $M \otimes A$ it suffices to prove that the second map in

$$M \otimes A \xrightarrow{\nu} (M \otimes A)/I \xrightarrow{\overline{\beta \otimes \mathrm{id}}} N \otimes A$$

is invertible ($I = \mathrm{im}(\alpha \otimes \mathrm{id})$). To construct an inverse $\overline{\psi}$, define $\psi \colon N \times A \to (M \otimes A)/I$ by $\psi(n \otimes a) = m \otimes a + I$ for any $m \in \beta^{-1}(n)$.

Right Exactness of $\otimes$

Thm. Tensor functors are right exact.

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is invertible ($I = \mathrm{im}(\alpha \otimes \mathrm{id})$). To construct an inverse $\overline{\psi}$, define $\psi\colon N \times A \to (M \otimes A)/I$ by $\psi(n \otimes a) = m \otimes a + I$ for any $m \in \beta^{-1}(n)$. ψ is well-defined because $\ker(\beta) = \mathrm{im}(\alpha)$ and $\mathrm{im}(\alpha) \otimes A = I$. ψ is bilinear, hence extends to a map $\overline{\psi}$ of $N \otimes A$. Since $\overline{\psi} \circ \overline{\beta \otimes \mathrm{id}}$ is the identity on elements $m \otimes a + I$, the map $\overline{\beta \otimes \mathrm{id}}$ is 1-1.

Nonexactness of $\otimes$

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Thm. If $S \subseteq R$ is a multiplicatively closed subset, then $S^{-1}R$ is flat as an R-module.

Eilenberg-Watts Theorem

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J. Indian Math. Soc. (N.S.) 24 (1960), 231–234 (1961).

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Theorem. If $F : R\text{-Mod} \to S\text{-Mod}$ is an additive functor, then the following are equivalent:

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