

Chain complexes, exact sequences

Chain complexes

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Chain complexes of R -modules form a category where a morphism $\alpha: K \rightarrow L$ is an indexed family of R -linear maps such that all squares are commutative in:

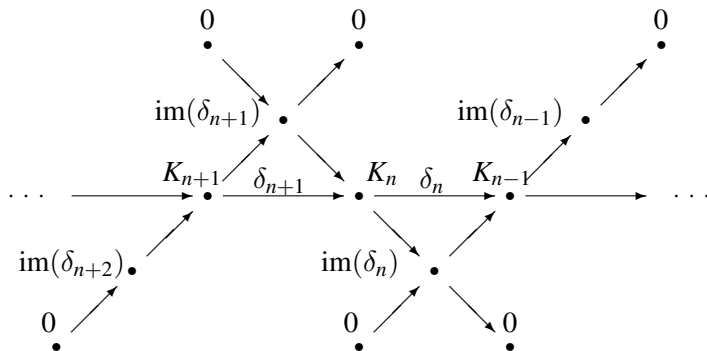
$$\begin{array}{ccccccc} \cdots & \xrightarrow{\delta_{n+1}} & K_n & \xrightarrow{\delta_n} & K_{n-1} & \xrightarrow{\delta_{n-1}} & \cdots \\ & & \alpha_n \downarrow & & \alpha_{n-1} \downarrow & & \\ \cdots & \xrightarrow{\varepsilon_{n+1}} & L_n & \xrightarrow{\varepsilon_n} & L_{n-1} & \xrightarrow{\varepsilon_{n-1}} & \cdots \end{array}$$

Factoring a Complex

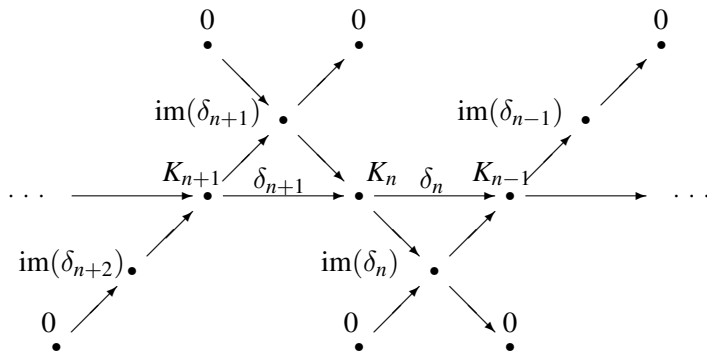
$$\cdots \longrightarrow \bullet \xrightarrow{\delta_{n+1}} \bullet \xrightarrow{\delta_n} \bullet \longrightarrow \cdots$$

The diagram illustrates a segment of a chain complex. It consists of three dots on the left, followed by a horizontal arrow pointing to a black dot. Above this arrow is the label K_{n+1} . From this dot, a second horizontal arrow points to another black dot. Above this arrow is the label δ_{n+1} . From this second dot, a third horizontal arrow points to a third black dot. Above this arrow is the label K_n . From this third dot, a fourth horizontal arrow points to a fourth black dot. Above this arrow is the label δ_n . From this fourth dot, a fifth horizontal arrow points to a fifth black dot. Above this arrow is the label K_{n-1} . Finally, a sixth horizontal arrow points to three dots on the right.

Factoring a Complex

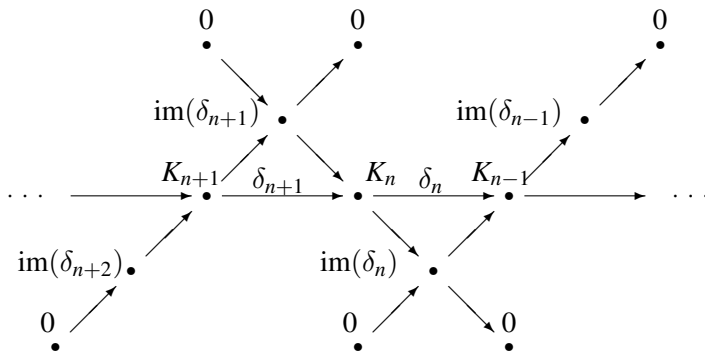


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$$0 \longrightarrow \text{im}(\delta_{n+1}) \xrightarrow{\subseteq} K_n \xrightarrow{\delta_n} \text{im}(\delta_n) \rightarrow 0$$

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This short sequence is exact except possibly at the middle, where the homology is the same as in the original sequence.

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For last item, write $x \in B$, as $x_{\text{im}(\alpha)} + x_{\text{im}(\delta)}$. Apply $\alpha\gamma + \delta\beta$ and show that x is unmoved.

Additive Functors

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Thm. Injectivity test: (Baer's Criterion) $\text{Hom}(_, Q)$ is exact on all SESs iff it is exact on those of the form $0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0$.

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is invertible ($I = \text{im}(\alpha \otimes \text{id})$). To construct an inverse $\overline{\psi}$, define $\psi: N \times A \rightarrow (M \otimes A)/I$ by $\psi(n \otimes a) = m \otimes a + I$ for any $m \in \beta^{-1}(n)$. ψ is well-defined because $\ker(\beta) = \text{im}(\alpha)$ and $\text{im}(\alpha) \otimes A = I$.

Right Exactness of \otimes

Thm. Tensor functors are right exact.

Proof: Assume $L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$ is exact, and consider

$$L \otimes A \xrightarrow{\alpha \otimes \text{id}} M \otimes A \xrightarrow{\beta \otimes \text{id}} N \otimes A \longrightarrow 0.$$

- $\beta \otimes \text{id}$ maps generators onto generators, so it is surjective.
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Thm. If $S \subseteq R$ is a multiplicatively closed subset, then $S^{-1}R$ is flat as an R -module.

Eilenberg-Watts Theorem

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Charles E. Watts, *Intrinsic characterizations of some additive functors.*

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Theorem. If $F : R\text{-Mod} \rightarrow S\text{-Mod}$ is an additive functor, then the following are equivalent:

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