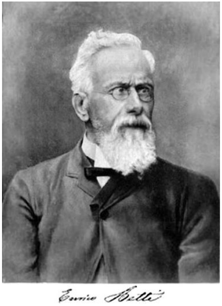
Homology and Cohomology

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8. These groups are invariants, which can be used for making distinctions.

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[Whiteboard!]

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$$\begin{aligned} \cdots \to \mathrm{Tor}_2^R(A, N) \to \quad & \mathrm{Tor}_1^R(A, L) \to \mathrm{Tor}_1^R(A, M) \to \\ \to \mathrm{Tor}_1^R(A, N) \quad & \to A \otimes_R L \to A \otimes_R M \to A \otimes_R N \to 0. \end{aligned}$$

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    3. The projective dimensions of M and N sum to the projective dimension of $M \otimes_R N$, and this is less than the Krull dimension of R.

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Exercise.

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$\mathrm{Ext}^n_{\mathbb{Z}}(A,B)$ can be interpreted as the groups of inequivalent n-extensions: $0 \to B \to E_n \to \cdots \to E_1 \to A \to 0$.