Compactness, connectedness and continuity.

Sometimes several properties can be shown to be equivalent, perhaps under extra assumptions or in restricted settings. Typically one of them is then taken to be the definition of the concept and the others are taken to be characterizations of the concept. Here are some examples.

Compactness.

(a) (Limit characterization, equivalent to compactness for space $\mathbb{R}$) $A \subseteq \mathbb{R}$ is *compact* every every sequence in A has a convergent subsequence whose limit is in A.

(b) (Metric characterization, equivalent to (a) for any metric space) $A \subseteq \mathbb{R}$ is *compact* if it is closed and bounded.

(c) (The Alexandrov-Urysohn definition. THE definition for any topological space) $A \subseteq X$ is *compact* if every open cover has a finite subcover.

Remarks about Compactness.

(i) The implication (b)$\Rightarrow$(c) is called the Heine-Borel Theorem. It holds for the metric space $\mathbb{R}$, and also for $\mathbb{R}^n$ with the ℓ_2-metric.

(ii) The Heine-Borel Theorem does not hold for every metric space. For example, $\mathbb{Q}$ is a metric subspace of $\mathbb{R}$, and the set $A = [0,1]_{\mathbb{Q}} \subseteq \mathbb{Q}$ is closed and bounded but not compact. The metric spaces for which (b)$\Rightarrow$(c) are said to have the "Heine-Borel Property".

(iii) Examples and nonexamples:
  (I) Any finite set is compact, including $\emptyset$.
  (II) $[0,1]_{\mathbb{R}}$ is compact.
  (III) The Cantor set is compact.
  (IV) $[0,1)$, $[0,\infty)$, $\mathbb{Q}$ all fail to be compact in $\mathbb{R}$.

Connectedness.

(a) (Characterization of connectedness in $\mathbb{R}$) $A \subseteq \mathbb{R}$ is *connected* if it is an interval. (This includes all sets of the form $(a,b), (a,b], [a,b)$, or $[a,b]$, where we allow a or b to be $\pm\infty$.)

(b) (THE definition, valid in any topological space) A subset $A \subseteq X$ is connected if it is not disconnected.

$A \subseteq X$ is *disconnected* if $A = A_1 \cup A_2$, where $A_1 \neq \emptyset \neq A_2$ and there exist open sets O_1, O_2 such that $O_1 \cap A = A_1$ and $O_2 \cap A = A_2$.

Test your intuition. Are the following subsets of $\mathbb{R}$ connected?

(I) $[0, 1) \cup [1, 2)$?
(II) $[0, 1) \cup (1, 2]$?
(III) The Cantor set?
(IV) $\mathbb{Q}$? $\mathbb{R}$? $\emptyset$?

Continuity.

(a) (Limit characterization) $f : X \to Y$ is continuous if $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$. ($f$ commutes with limits.)

(b) (Metric characterization, equivalent to (a) for any metric space) $f : X \to Y$ is continuous at x_0 if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)((d_X(x, x_0) < \delta) \to (d_Y(f(x), f(x_0)) < \varepsilon)).$$

f is continuous if it is continuous at every $x_0 \in X$.

If $X = Y = \mathbb{R}$ we write

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)((|x - x_0| < \delta) \to (|f(x) - f(x_0)| < \varepsilon)).$$

(c) (THE definition for any topological space) A function $f : X \to Y$ is continuous if the inverse image of any open set is open. (That is, $f^{-1}(O)$ is open.)

Some theorems.

(I) (Cantor Intersection Theorem) This is just the NIP with closed bounded intervals replaced by compact sets.
(II) The continuous image of a compact set is compact.
(III) The continuous image of a connected set is connected.
(IV) The continuous image of a closed bounded interval is a closed bounded interval.
(V) (Extreme Value Theorem – due to Bolzano) A continuous function on a compact set attains max amd min values on that set.
(VI) (Bolzano's Theorem. Same as Intermediate Value Theorem) If f is continuous on $[a, b]$, then f assumes every intermediate value between $f(a)$ and $f(b)$.