SET THEORY (MATH 4730)

SUMMARY OF TOPICS FROM 10/14/19-12/9/19

VI. Ordinals.
- (a) Well-ordered sets. (Definition. Uniqueness properties from Section 6.1.3.)
- (b) Ordinal numbers. (Definition. Examples. Successor vs. limit ordinals. Ordinals are well-ordered. Order type. Natural numbers are exactly the finite ordinals. $\alpha \notin \alpha$ provable without Foundation.)
- (c) Axiom of Replacement.
- (d) Paradoxes of set theory: Russell's Paradox, Cantor's Paradox, Burali-Forti Paradox.
- (e) Transfinite induction.
- (f) Recursion Theorem.
- (g) Ordinal arithmetic. (Definitions, properties, order-theoretic interpretations.)

VII. Cardinals.
- (a) Definition (= initial ordinal) and notation.
- (b) Hartogs number of a set.
- (c) Definitions for cardinal arithmetic.
- (d) Canonical well-ordering of $\omega_\alpha \times \omega_\alpha$ has order-type ω_α.
- (e) $\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \max(\aleph_\alpha, \aleph_\beta)$.

VIII. Axiom of Choice.
- (a) ZF proves that A can be well-ordered iff $\mathcal{P}(A)$ has a choice function.
- (b) Within ZF, AC is equivalent to:
  - (i) Well-Ordering Theorem.
  - (ii) Zorn's Lemma.
  - (iii) Every surjective function has a right inverse.
  - (iv) Every set is equipotent with a unique initial ordinal.
  - (v) Any two sets have comparable cardinalities.
  - (vi) $|A \times A| = |A|$ for any infinite A.
  - (vii) Every vector space has a basis.
- (c) AC implies that
  - (i) every infinite set has a countably infinite subset.
  - (ii) every Dedekind finite set is finite.
  - (iii) a countable union of countable sets is countable.
  - (iv) $2^{\aleph_0} \geq \aleph_1$.

IX. Cardinal Arithmetic.
- (a) Cofinality.
- (b) Every infinite cardinal λ can be partitioned into λ-many cofinal λ-sequences.
- (c) Infinite sums and products.

(d) Kőnig's Theorem (+ the corollary $\kappa^{\mathrm{cf}(\kappa)} > \kappa$).
(e) Regular and singular cardinals.
(f) Successor cardinals are regular. $\aleph_\omega$ and $\beth_\omega$ are singular.
(g) Within ZFC a cardinal κ may equal $|\mathbb{R}| = |\mathcal{P}(\omega)| = 2^{\aleph_0}$ iff $\mathrm{cf}(\kappa)$ is uncountable.
(h) Cardinal exponentiation.
(i) Statements of the CH and the GCH.

General advice on preparing for a math test.
Be prepared to demonstrate understanding in the following ways.

(i) Know the definitions of new concepts, and the meanings of the definitions.
(ii) Know the statements and meanings of the major theorems.
(iii) Know examples/counterexamples. (The purpose of an example is to illustrate the extent of a definition or theorem. The purpose of a counterexample is to indicate the limits of a definition or theorem.)
(iv) Know how to perform the different kinds of calculations discussed in class.
(v) Be prepared to prove elementary statements. (Understanding the proofs done in class is the best preparation for this.)
(vi) Know how to correct mistakes made on old HW.

Sample Problems.

(1) (a) Define "well-ordered set".
  (b) Define "ordinal".
  (c) Show that 0 is an ordinal and that the successor of an ordinal is an ordinal.
  (d) Show that an element of an ordinal is an ordinal.

(2) What are
  (a) Russell's Paradox?
  (b) Cantor's Paradox?
  (c) The Burali-Forti Paradox?

(3) Explain why the class of all ordinals is not a set.

(4) Is the set of proper subsets of ω, ordered by inclusion, an inductively ordered set? What conclusions, if any, can be drawn?

(5) If the poset $\langle P; \leq \rangle$ is inductively ordered, and $X \subseteq P$, must the subposet $\langle X; \leq \rangle$ be inductively ordered?

(6) Explain why the Hartogs number of a set is a cardinal.

(7) Explain why the cofinality of an ordinal is a cardinal.

(8) Show that every vector space has a basis.

(9) Show that if κ is an infinite cardinal, then $2^\kappa = \kappa^\kappa$.

(10) Show that $\mathrm{cf}\,(2^\kappa) > \kappa$.