SET THEORY

SUMMARY OF TOPICS FROM 8/26/15-10/7/15

I. Axiomatic Set Theory
   (a) Extensionality.
   (b) Valid constructions of new sets (pairing, union, power set, comprehension, replacement, choice).
   (c) Empty set, successor of a set.
   (d) The Axiom of Infinity.
   (e) Intersection versus Union.
   (f) The directed graph models of set theory.
   (g) Naive set theory is inconsistent (Russell's Paradox).

II. Relations and Functions
   (a) Ordered pairs (Kuratowski encoding), Cartesian products.
   (b) Relations.
   (c) Functions, equivalence relations, partitions, kernel, image, coimage.
   (d) Ordered sets, linear orders, well-orders.

III. Natural numbers
   (a) Definition of ω.
   (b) Induction.
   (c) ω is well-ordered.
   (d) Recursion Theorem.
   (e) Arithmetic of ω.

IV. Cardinality of sets
   (a) Equipotence ($|A| = |B|$). $|A| \leq |B|$, $|A| \leq m$, etc.
   (b) Cantor-Bernstein-Schröder Theorem.
   (c) Finite sets and their properties.
   (d) Countable sets.

V. Ordinals and cardinals.
   (a) Transitive sets.
   (b) Definition of ordinal and cardinal number.
   (c) Equipotence classes of infinite ordinals are bounded intervals.

General advice on preparing for a math test.

Be prepared to demonstrate understanding in the following ways.

(i) Know the definitions of new concepts, and the meanings of the definitions.
(ii) Know the statements and meanings of the major theorems.

(iii) Know examples/counterexamples. (The purpose of an example is to illustrate the extent of a definition or theorem. The purpose of a counterexample is to indicate the limits of a definition or theorem.)
(iv) Know how to perform the different kinds of calculations discussed in class.
(v) Be prepared to prove elementary statements. (Understanding the proofs done in class is the best preparation for this.)
(vi) Know how to correct mistakes made on old HW.

Sample Problems.

(1) Why is naive set theory inconsistent?
(2) Write the following axioms of set theory formally: Empty Set, Extensionality, and Union.
(3) Explain why the intersection of two sets is a set.
(4) Prove or disprove:
 (a) $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$.
 (b) $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$.
(5) Define: inductive set, function, maximal element of a poset, equipotence.
(6) What is the kernel of the successor function? What is the image of the successor function? What is the coimage of the successor function?
(7) Explain why induction is a valid form of proof. (Your explanation should make use of the fact that ω is a subset of every inductive set.)
(8) Show that $m + n = 0$ implies $m = n = 0$ for natural numbers m and n. Then show that $m + n = 1$ implies that m and n are 0 and 1 in some order.
(9) Show that the intersection of two transitive sets is transitive. (What about union?)
(10) Suppose I have a set of pairwise disjoint circles in the plane, all of radius 1. Explain why my set is at most countable.
(11) Let P be the set of partial orderings of ω, let L be the set of all linear orderings of ω, and let W be the set of all well-orderings of ω. How are the cardinalities of the sets P, L, W related to the cardinalities of ω and $\mathcal{P}(\omega)$?